Waves with constant phase velocity in relativistic plasmas

Thomas Pesch

Institut für Theoretische Physik A
RWTH Aachen

ILIAS-Arbeitsseminar

Darmstadt, 19.9.2006

RWTHAACHEN UNIVERSITY

Overview

- Introduction
- Cold plasma dynamics - Basic equations
- Circular polarization
- Linear polarization:
  - Small plasma density
  - Critical plasma density
  - Intermediate plasma density

RWTH AACHEN UNIVERSITY

Electromagnetic waves in plamas

Relativistic treatment:

- Relativistic particle motion in an electromagnetic wave
  (Landau, Lifschitz; Sarachik, Schappert)
- Relativistic self-focusing in weak relativistic plasma
  (Max, Arons, Langdon 1974)
- Particle-in-cell simulations
  (Pukhov 1999)
- Selfconsistent waves of constant phase velocity in plasmas
  (Alkhiezer, Polovin 1956; Kaw, Dawson 1970)

Goal

Analytical investigation of electromagnetic plane waves with a constant phase velocity in a homogen cold relativistic plasmas.

$\rightarrow$ dispersion relation, trajectory; polarization, density

Assumptions:

- $\phi = \omega t - kx = t - \frac{1}{v_{\mathrm{ph}}} x, \qquad f(\boldsymbol{x}, t) = f(\phi)$
- cold plasma
- ion motion is negligible

Akhiezer, Polovin 1956
Kaw, Dawson 1970
...

Relativistic cold plasma fluid dynamics

- Charge- and currentdensity:

$$\rho = -e\,(n_\mathrm{e} - Zn_\mathrm{i})$$
$$\boldsymbol{j} = -e\,(n_\mathrm{e}\boldsymbol{v}_\mathrm{e} - Zn_\mathrm{i}\boldsymbol{v}_\mathrm{i})$$

- Maxwell equation´s:

$$\begin{aligned}
\nabla \times \boldsymbol{B} - \partial_\mathrm{t}\boldsymbol{E} &= -\omega_\mathrm{p}^2 n\boldsymbol{v} \\
\nabla \times \boldsymbol{E} + \partial_\mathrm{t}\boldsymbol{B} &= 0 \\
\nabla \cdot \boldsymbol{E} &= \omega_\mathrm{p}^2(1-n) \\
\nabla \cdot \boldsymbol{B} &= 0
\end{aligned}$$

$$n = \frac{n_\mathrm{e}}{n_0}$$

$$\omega_\mathrm{p}^2 = \frac{4\pi e^2 n_0}{m}$$

- Lorentz equation:

$$\frac{d}{dt}(\gamma\boldsymbol{v}) = -(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})$$

RWTH AACHEN UNIVERSITY

Plane waves of constant phase velocity

$$\phi = t - \frac{1}{v_{\rm ph}} x, \qquad f(\boldsymbol{x}, t) = f(\phi)$$

- Propagation in x-direction:

$$\boldsymbol{E} \;=\; (E_{\rm x}, E_{\rm y}, E_{\rm z}), \qquad \boldsymbol{B} \;=\; (0, B_{\rm y}, B_{\rm z})$$

- Substitution of derivations:

proper time

$$\partial_{\rm t} \longrightarrow \frac{d}{d\phi} \longrightarrow \frac{v_{\rm ph}}{v_{\rm ph}\gamma - p_{\rm x}} \frac{d}{d\tau}$$

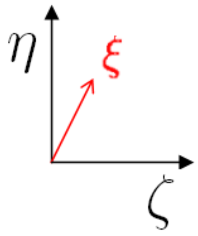
$$\partial_{\rm x} \longrightarrow -\frac{1}{v_{\rm ph}} \frac{d}{d\phi} \longrightarrow \frac{-1}{v_{\rm ph}\gamma - p_{\rm x}} \frac{d}{d\tau}$$

Akhiezer, Polovin 1956
Kaw, Dawson 1970
…

Simplified set of equations

Lagrange-coordinates: $\xi = x - x_0, \quad \eta = y - y_0, \quad \zeta = z - z_0$

Set of nonlinear ordinary differentialequations:



$$\begin{aligned} \ddot{\xi} + \omega_{\mathrm{p}}^2 \gamma \xi &= -\sigma^2 \eta p_{\mathrm{y}} - \sigma^2 \zeta p_{\mathrm{z}} \\ \ddot{\eta} + \sigma^2 v_{\mathrm{ph}} \gamma \eta &= p_{\mathrm{x}} \eta \\ \ddot{\zeta} + \sigma^2 v_{\mathrm{ph}} \gamma \zeta &= p_{\mathrm{x}} \zeta \end{aligned}$$

$$\frac{d}{d\tau}$$

„just“ coupled relativistic harmonic oscillators!

$$\sigma^2 = \frac{\omega_{\mathrm{p}}^2 v_{\mathrm{ph}}}{v_{\mathrm{ph}}^2 - 1}$$

$$\begin{aligned} p_{\mathrm{x}} &= \dot{\xi}, \ldots \\ \gamma &= \sqrt{1 + p_{\mathrm{x}}^2 + p_{\mathrm{y}}^2 + p_{\mathrm{z}}^2} \end{aligned}$$

RWTH AACHEN UNIVERSITY

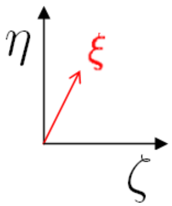
Simplified set of equations

$$\begin{aligned} E_{\mathrm{x}} &= \omega_{\mathrm{p}}^2 \xi \\ E_{\mathrm{y}} &= v_{\mathrm{ph}} B_{\mathrm{z}}, & B_{\mathrm{z}} &= \sigma^2 \eta \\ E_{\mathrm{z}} &= -v_{\mathrm{ph}} B_{\mathrm{y}}, & B_{\mathrm{y}} &= -\sigma^2 \zeta \end{aligned}$$

$$n = \frac{v_{\mathrm{ph}}}{v_{\mathrm{ph}} - v_{\mathrm{x}}}$$

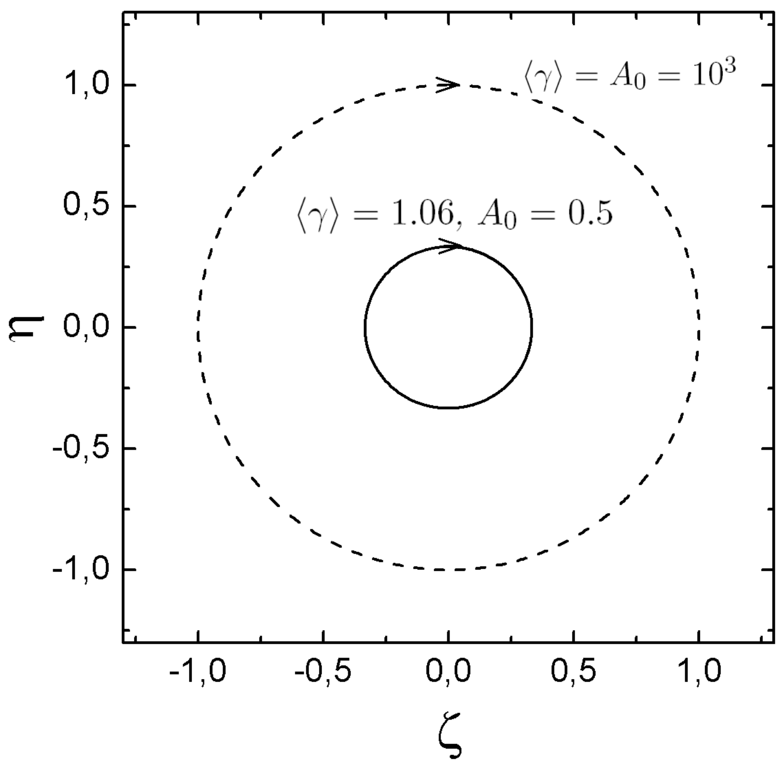
Relativistic vacuum solutions

Relativistic particle motion in an electromagnetic wave.



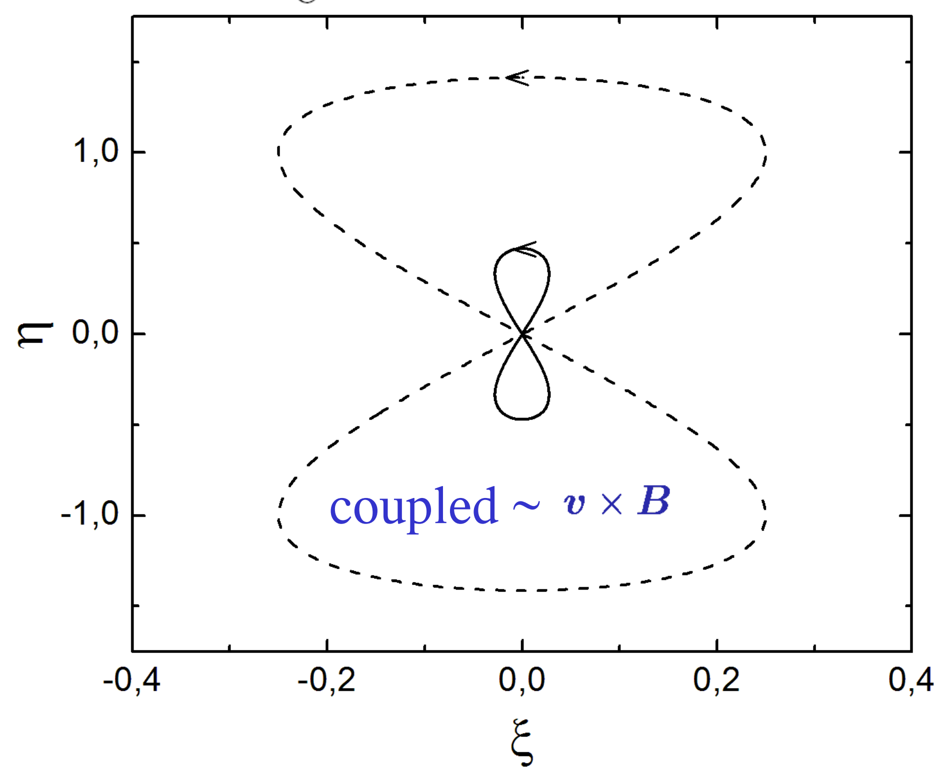
- Circular polarization: $\xi = 0$

$$\zeta = \eta_0 \sin(\phi), \;\; \eta = \pm\eta_0 \cos(\phi)$$



- Linear polarization: $\zeta = 0$

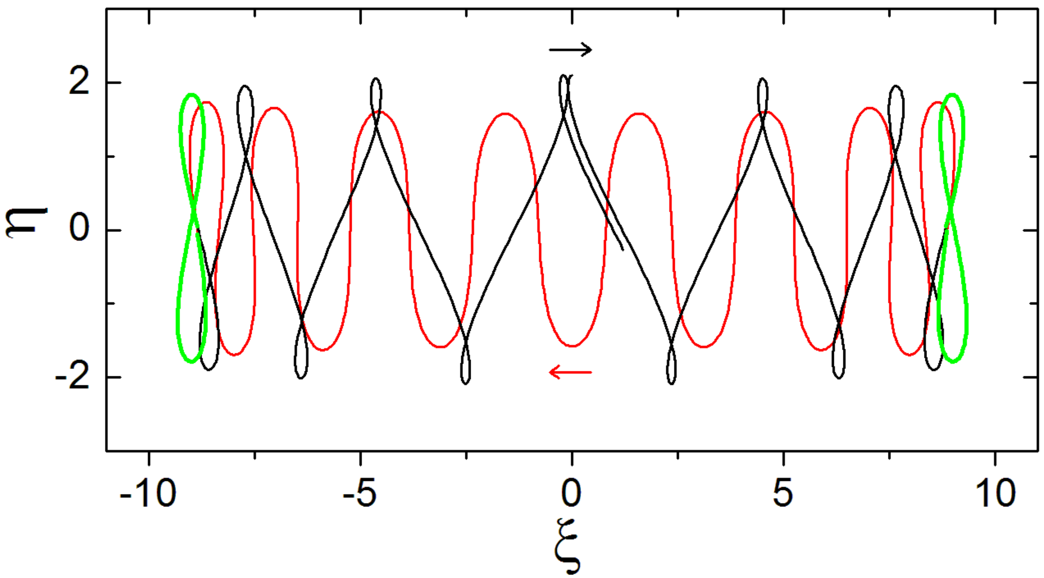
$$\xi = -\frac{1}{8}\eta_0^2 \sin(2\phi), \;\; \eta = \eta_0 \cos(\phi)$$

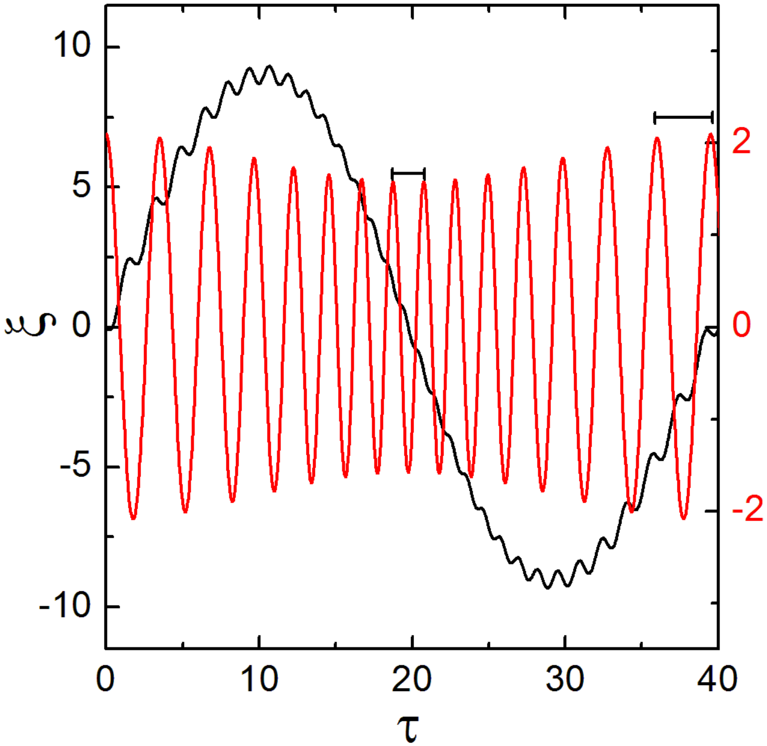


Relativistic plasma solutions

Relativistic plasma: Electromagnetic wave couples nonlinear with electorstatic oscillation.

$\longrightarrow$ Selfmodulation of amplitude and phase
$\xrightarrow{\text{i.g.}}$ Nonperiodic solution





Linear polarization $\omega_p = 0.1, \quad p_{x0} = -0.6, \quad \eta_0 = 2.1$

Periodic solutions

Generalized vacuum trajectories

- Closed particle trajectories
- Only one frequency (in rest frame: $\Omega = \langle\gamma\rangle\,\omega$)
- No selfmodulation, but still nonlinear coupling
- One inital condition has to be adjusted

$\rightarrow$ Analytic investigation is possible

Circular polarization

$$\xi = 0$$

$$\ddot{\eta} + \sigma^2 v_{\rm ph} \gamma \eta = 0$$

$$\ddot{\zeta} + \sigma^2 v_{\rm ph} \gamma \zeta = 0$$

exact solution

$$\eta = \pm \eta_0 \cos(\phi_\tau)$$

$$\zeta = \eta_0 \sin(\phi_\tau)$$

$$\phi_\tau = \Omega \tau$$

- Equal to vacuum solution
- Circular electron current, no density bunching

Phasevelocity:

$$v_{\rm ph} = \frac{1}{\sqrt{1 - \frac{\omega_{\rm p}^2}{\langle \gamma \rangle}}}$$

$\Longleftrightarrow$ $v_{\rm ph} = \frac{\omega}{k}$

Dispersion relation:

$$k^2 = \omega^2 - \frac{\omega_{\rm p}^2}{\langle \gamma \rangle}$$

$$\gamma = \langle \gamma \rangle = \frac{1}{\sqrt{1 - \eta_0^2}}$$

Akhiezer, Polovin

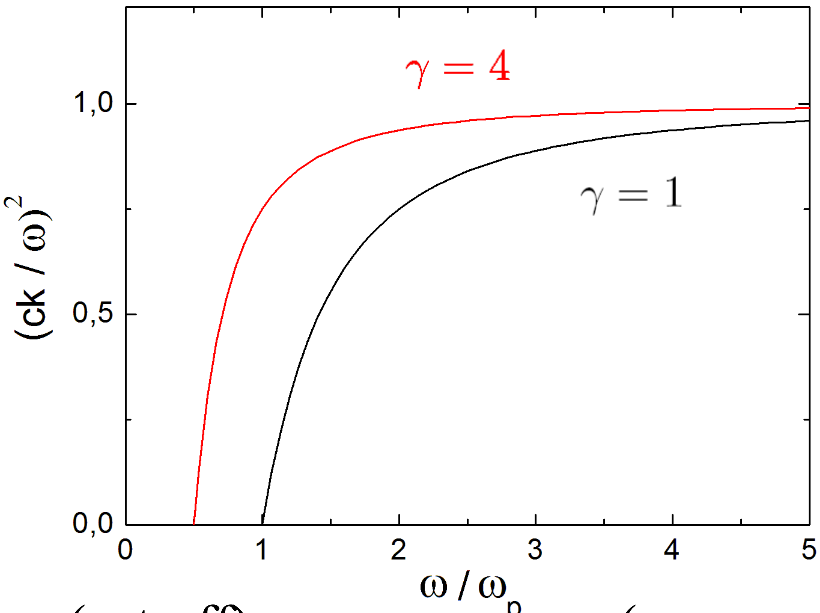
Circular polarization

Dispersion relation:

$$k^2 = \omega^2 - \frac{\omega_p^2}{\langle\gamma\rangle}$$

nonrelativistic

$$k^2 = \omega^2 - \omega_p^2$$



(cut-off)

$$\omega \longrightarrow \omega_{\mathrm{p\ eff.}}$$

(vacuum-limit)

$$\omega \gg \omega_p$$

$$\frac{\omega_p^2}{\langle\gamma\rangle} = \frac{4\pi e^2 n_0}{\langle\gamma\rangle\, m} = \omega_{\mathrm{p\ eff.}}^2$$

- relativistic mass increase
- effective plasma frequency

RWTHAACHEN UNIVERSITY

Linear polarization

$$\zeta = 0$$

$$\ddot{\xi} + \omega_{\mathrm{p}}^2 \gamma \xi = -\sigma^2 \eta p_{\mathrm{y}}$$

$$\ddot{\eta} + \sigma^2 (v_{\mathrm{ph}} \gamma - p_{\mathrm{x}}) \eta = 0$$

No exact solution ...

Small plasma density

Critical plasma density

Small plasma density

Small density is pertubation of vacuum solution.

$$\begin{aligned} \zeta &= 0 \\ \ddot{\xi} + \omega_{\rm p}^2 \gamma \xi &= -\sigma^2 \eta p_{\rm y} \\ \ddot{\eta} + \sigma^2 (v_{\rm ph}\gamma - p_{\rm x})\eta &= 0 \end{aligned}$$

No exact solution - expansion in $\omega_{\rm p}^2 \sim n_0$

$$\begin{aligned} \xi &= \hat{\eta}\cos(\phi_\tau) + \omega_{\rm p}^2 \xi^{(1)} + \mathcal{O}(\omega_{\rm p}^4) \\ \eta &= -\frac{1}{8}\hat{\eta}^2 \sin(2\phi_\tau) + \omega_{\rm p}^2 \eta^{(1)} + \mathcal{O}(\omega_{\rm p}^4) \end{aligned}$$

Small plasma density

No exact solution - expansion in $\omega_\mathrm{p}^2 \sim n_0$ →

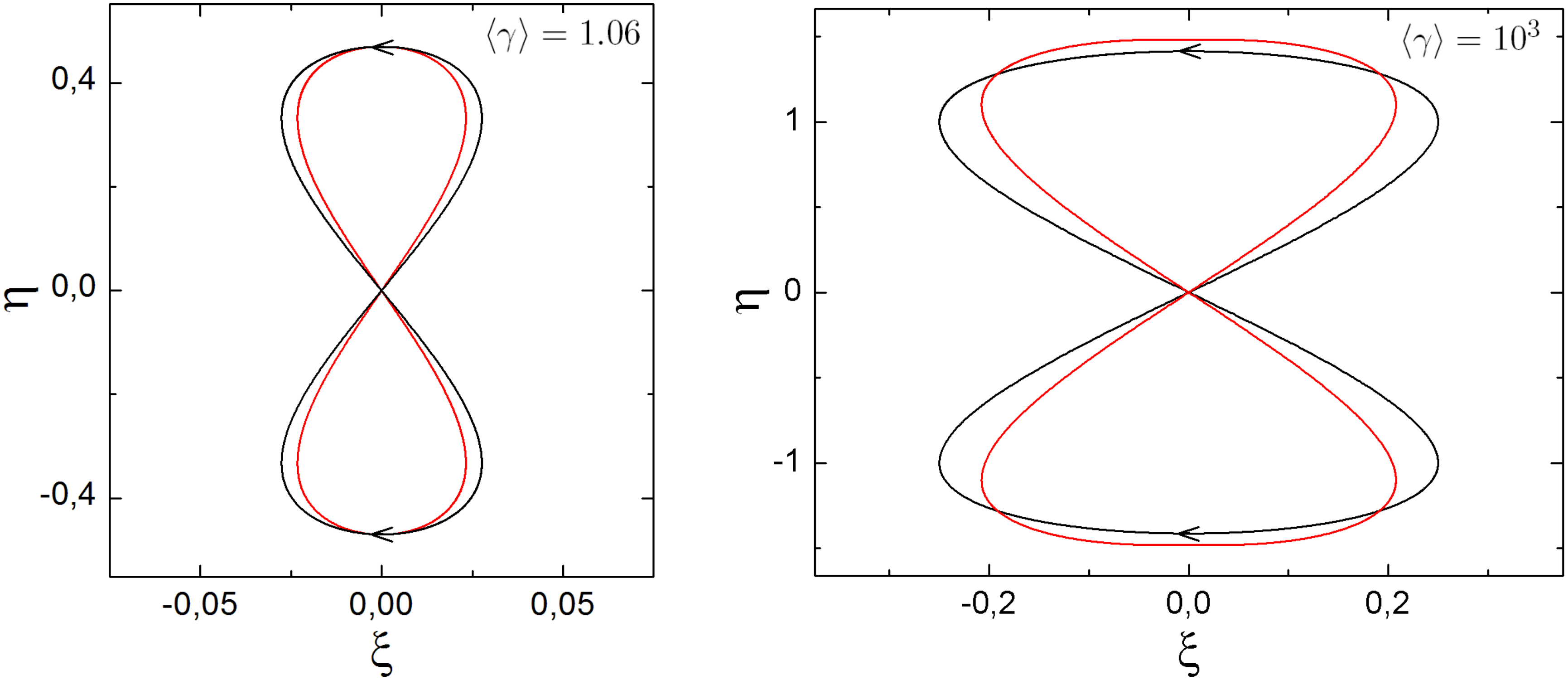
$$\eta = \hat{\eta}\left(1 + \frac{\omega_\mathrm{p}^2}{\langle\gamma\rangle}\frac{1}{64}\hat{\eta}^2\right)\cos(\phi_\tau) - \frac{\omega_\mathrm{p}^2}{\langle\gamma\rangle}\frac{3}{256}\hat{\eta}^3\cos(3\phi_\tau) + \mathcal{O}(\omega_\mathrm{p}^4)$$

$$\xi = -\frac{1}{8}\hat{\eta}^2\left(1 - \frac{\omega_\mathrm{p}^2}{\langle\gamma\rangle}\frac{1}{4}(1 - \frac{13}{32}\hat{\eta}^2)\right)\sin(2\phi_\tau) + \frac{\omega_\mathrm{p}^2}{\langle\gamma\rangle}\frac{5}{256}\frac{1}{8}\hat{\eta}^4\sin(4\phi_\tau) + \mathcal{O}(\omega_\mathrm{p}^4)$$

$$\langle\gamma\rangle = \frac{1}{\sqrt{1 - \frac{1}{2}\hat{\eta}^2}}$$

- Higher harmonics (odd or even)
- Modified amplitude
- Density bunching $n = \dfrac{v_\mathrm{ph}}{v_\mathrm{ph} - v_\mathrm{x}} \neq const.$

Trajectory



$$\eta = \hat{\eta}\left(1 + \frac{\omega_p^2}{\langle\gamma\rangle}\frac{1}{64}\hat{\eta}^2\right)\cos(\phi_\tau) - \frac{\omega_p^2}{\langle\gamma\rangle}\frac{3}{256}\hat{\eta}^3\cos(3\phi_\tau) + \mathcal{O}(\omega_p^4)$$

$$\xi = -\frac{1}{8}\hat{\eta}^2\left(1 - \frac{\omega_p^2}{\langle\gamma\rangle}\frac{1}{4}(1 - \frac{13}{32}\hat{\eta}^2)\right)\sin(2\phi_\tau) + \frac{\omega_p^2}{\langle\gamma\rangle}\frac{5}{256}\frac{1}{8}\hat{\eta}^4\sin(4\phi_\tau) + \mathcal{O}(\omega_p^4)$$

Dispersion relation

Solution $\mathcal{O}(\omega_{\mathrm{p}}^2)$ → Dispersion relation $\mathcal{O}(\omega_{\mathrm{p}}^4)$

Phasevelocity:

$$v_{\mathrm{ph}} = 1 + \frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle} + \frac{3}{8}\left(1 - \frac{1}{8}\hat{\eta}^2\right)\frac{\omega_{\mathrm{p}}^4}{\langle\gamma\rangle^2} + \mathcal{O}(\omega_{\mathrm{p}}^6)$$

Dispersion relation:

$$k^2 = \omega^2\left(1 - \frac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle} + \frac{3}{32}\hat{\eta}^2\frac{\omega_{\mathrm{p}}^4}{\langle\gamma\rangle^2}\right) + \mathcal{O}(\omega_{\mathrm{p}}^6)$$

Difference between linear and circular polarization.

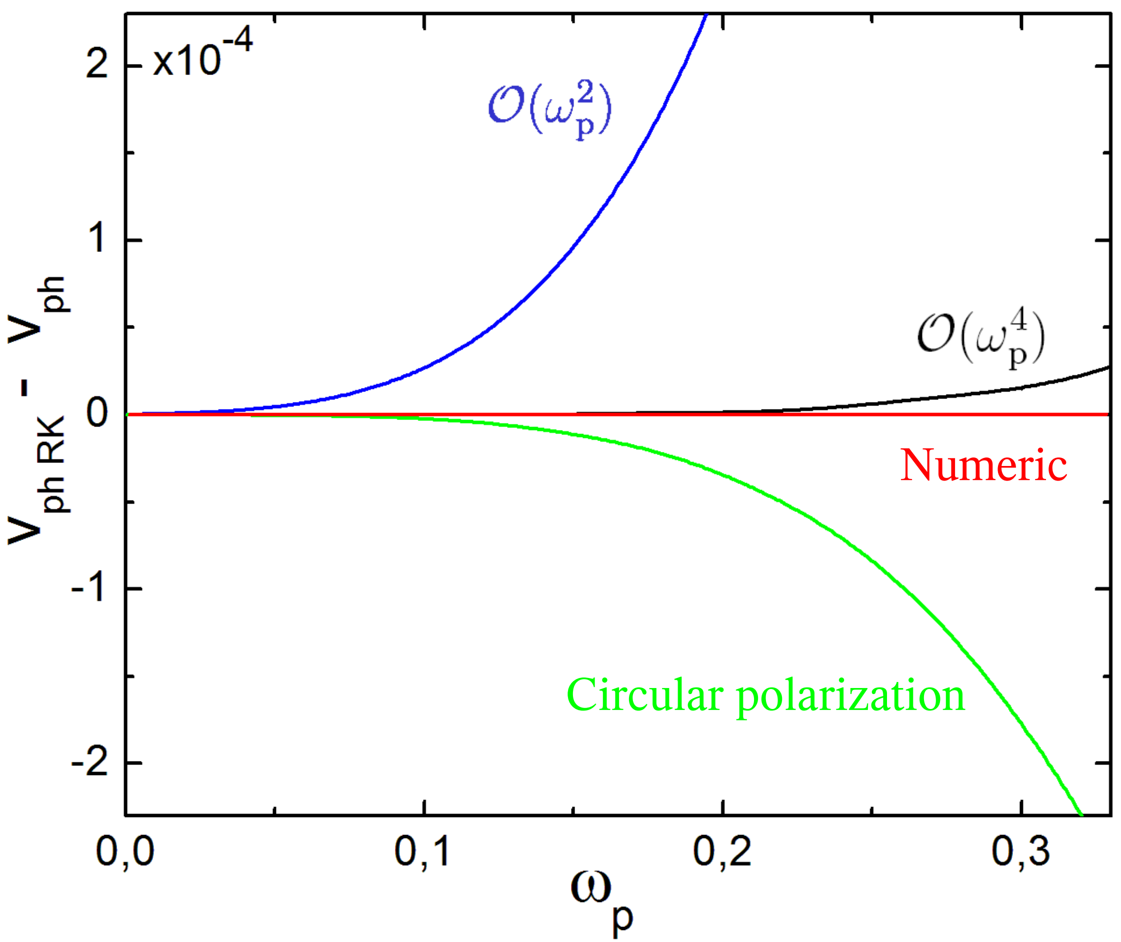
$$v_{\mathrm{ph}} = \frac{1}{\sqrt{1 - \frac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle}}} = 1 + \frac{1}{2}\frac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle} + \frac{3}{8}\frac{\omega_{\mathrm{p}}^4}{\langle\gamma\rangle^2} + \mathcal{O}(\omega_{\mathrm{p}}^6)$$

$$k^2 = \omega^2\left(1 - \frac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle}\right)$$ circular polarization

RWTH AACHEN UNIVERSITY

Runge-Kutta-Integration

Phasevelocity



Critical plasma density

- Circular polarization: $k^2 = \omega^2 - \dfrac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle}$

  Critical density $\frac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle} \to \omega^2 : \; k \to 0$

- Linear polarization:

  generalization of 2d eight-like solution $\neq$ 1d electrostatic oscillation:

$$\xi, \eta$$

$$\ddot{\xi} + \omega_{\mathrm{p}}^2/\gamma\,\xi = 0$$

RWTH AACHEN UNIVERSITY

Linear polarization

$$\ddot{\xi} + \omega_{\rm p}^2 \gamma \xi = -\frac{\omega_{\rm p}^2 k}{1-k^2} \eta p_{\rm y}$$

$$\ddot{\eta} + \frac{\omega_{\rm p}^2}{1-k^2} \gamma \eta = \frac{\omega_{\rm p}^2 k}{1-k^2} \eta p_{\rm x}$$

No exact solution - expansion in $k = \frac{1}{v_{\rm ph}}$ $(k \rightarrow 0)$

Linear polarization

$$\begin{aligned}\ddot{\xi} + \omega_{\rm p}^2 \gamma \xi &= -\frac{\omega_{\rm p}^2 k}{1-k^2}\eta p_{\rm y} \\ \ddot{\eta} + \frac{\omega_{\rm p}^2}{1-k^2}\gamma \eta &= \frac{\omega_{\rm p}^2 k}{1-k^2}\eta p_{\rm x}\end{aligned}$$

No exact solution - expansion in $k = \frac{1}{v_{\rm ph}}$ $(k \longrightarrow 0)$

Lowest Order:

$$\begin{aligned}\ddot{\xi} + \omega_{\rm p}^2 \gamma \xi &= 0 \\ \ddot{\eta} + \omega_{\rm p}^2 \gamma \eta &= 0\end{aligned} \longrightarrow \begin{aligned}\xi &= \eta_0 \sin(\phi_\tau) + \mathcal{O}(k) \\ \eta &= \pm\eta_0 \cos(\phi_\tau) + \mathcal{O}(k)\end{aligned}$$

Similar equations as for circular polarization.

Circular motion in longitudinal and transverse direction.

Two coupled electrostatic oscillations.

$\boldsymbol{B} = \mathcal{O}(k)$

RWTH AACHEN UNIVERSITY

Linear polarization

No exact solution - expansion in $k = \frac{1}{v_{\mathrm{ph}}}$

Higher Orders:

$$\begin{aligned} \xi &= \xi^{(0)} + k\xi^{(1)} + k^2\xi^{(2)} + \mathcal{O}(k^3) \\ \eta &= \eta^{(0)} + k\eta^{(1)} + k^2\eta^{(2)} + \mathcal{O}(k^3) \end{aligned}$$

$$\omega_{\mathrm{p}}^2 = \omega_{\mathrm{p}}^{2\,(0)} + k\omega_{\mathrm{p}}^{2\,(1)} + k^2\omega_{\mathrm{p}}^{2\,(2)} + \mathcal{O}(k^3)$$

1. 2. (Necessary for non-trivial dispersion relation)

$$\frac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle} = 1 - k^2 = 1 + \mathcal{O}(k^2)$$

circular polarization

RWTH AACHEN UNIVERSITY

Linear polarization

No exact solution - expansion in $k = \frac{1}{v_{\mathrm{ph}}}$ →

$$
\begin{aligned}
\xi &= \left(\hat{\eta} + k^2 \frac{8\hat{\eta}^6 - 19\hat{\eta}^4 + 99\hat{\eta}^2 - 144}{16\hat{\eta}(3-\hat{\eta}^2)^2}\right)\sin(\phi_\tau) \\
&\quad - k\frac{\hat{\eta}^2}{2(3-\hat{\eta}^2)}\sin(2\phi_\tau) - k^2\frac{9\hat{\eta}(1-\hat{\eta}^2)}{16(3-\hat{\eta}^2)^2}\sin(3\phi_\tau) + \mathcal{O}(k^3)
\end{aligned}
$$

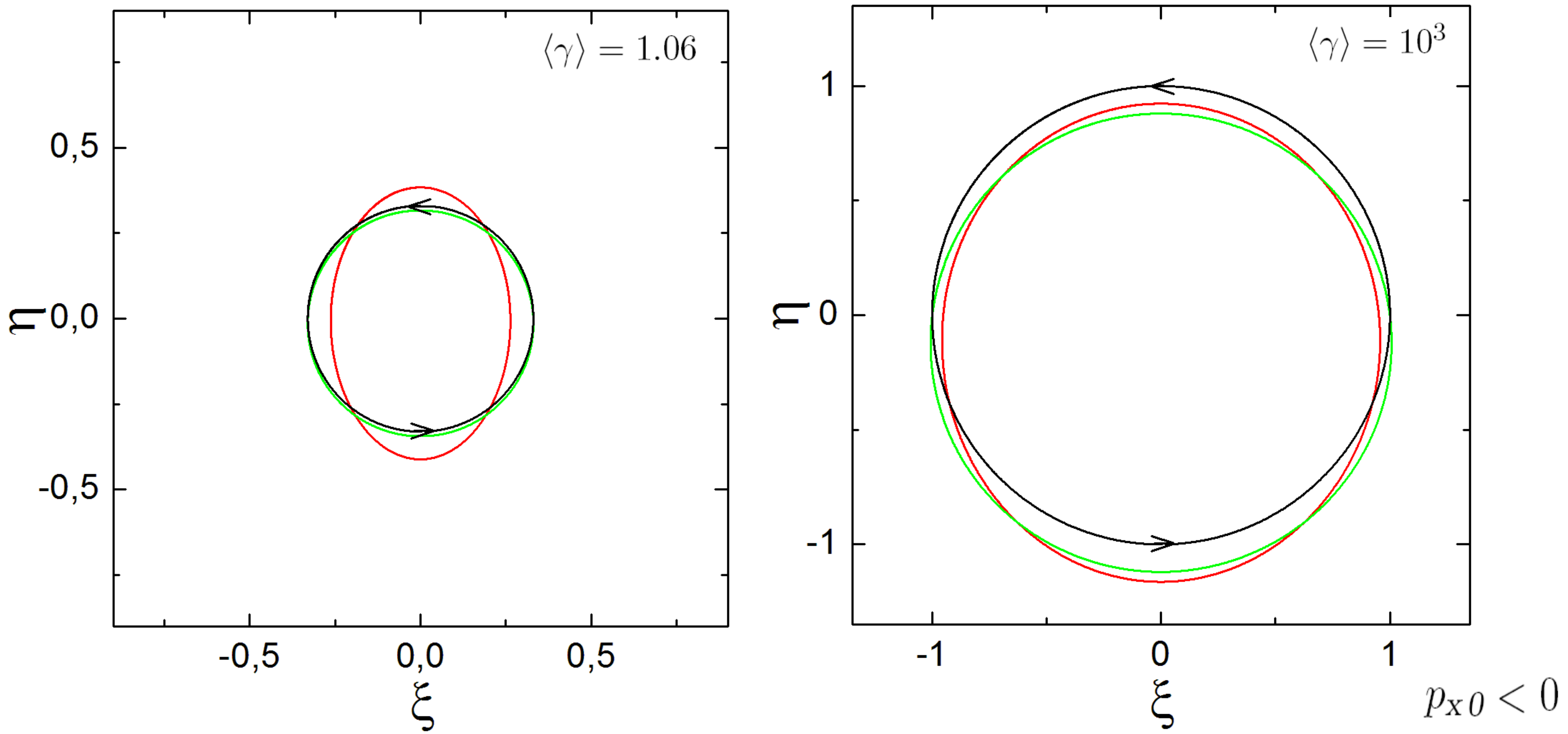
$$
\begin{aligned}
\eta &= \pm\Bigg(\left(\hat{\eta} + k^2\frac{3(\hat{\eta}^4 - 33\hat{\eta}^2 + 48)}{16\hat{\eta}(3-\hat{\eta}^2)^2}\right)\cos(\phi_\tau) \\
&\quad - k\frac{\hat{\eta}^2}{2(3-\hat{\eta}^2)}\cos(2\phi_\tau) - k^2\frac{9\hat{\eta}(1-\hat{\eta}^2)}{16(3-\hat{\eta}^2)^2}\cos(3\phi_\tau) \\
&\quad + k\frac{3\hat{\eta}^2}{2(3-\hat{\eta}^2)}\Bigg) + \mathcal{O}(k^3)
\end{aligned}
$$

$$\langle\gamma\rangle = \frac{1}{\sqrt{1-\hat{\eta}^2}}$$

- Higher harmonics (odd and even)
- Amplitude modification is second order effect
- Constant shift in transverse direction

RWTH AACHEN UNIVERSITY

Trajectory



Transverse shift caused by $\boldsymbol{v} \times \boldsymbol{B}$ -force: $B_\mathrm{z} = k\omega_\mathrm{p}^2\eta + \mathcal{O}(k^2)$

RWTHAACHEN UNIVERSITY

Dispersion relation

$$\frac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle} = \omega^2 - k^2 \frac{\hat{\eta}^4 - 6\hat{\eta}^2 + \frac{9}{2}}{(3-\hat{\eta}^2)^2} + \mathcal{O}(k^3)$$

$$= \omega^2 - k^2 \underbrace{\left(1 - \frac{9}{2(3-\hat{\eta}^2)^2}\right)}_{\substack{\to +\frac{1}{2} \quad (\langle\gamma\rangle \to 1) \\ \to -\frac{1}{8} \quad (\langle\gamma\rangle \to \infty)}} + \mathcal{O}(k^3)$$

Difference between linear and circular polarization.

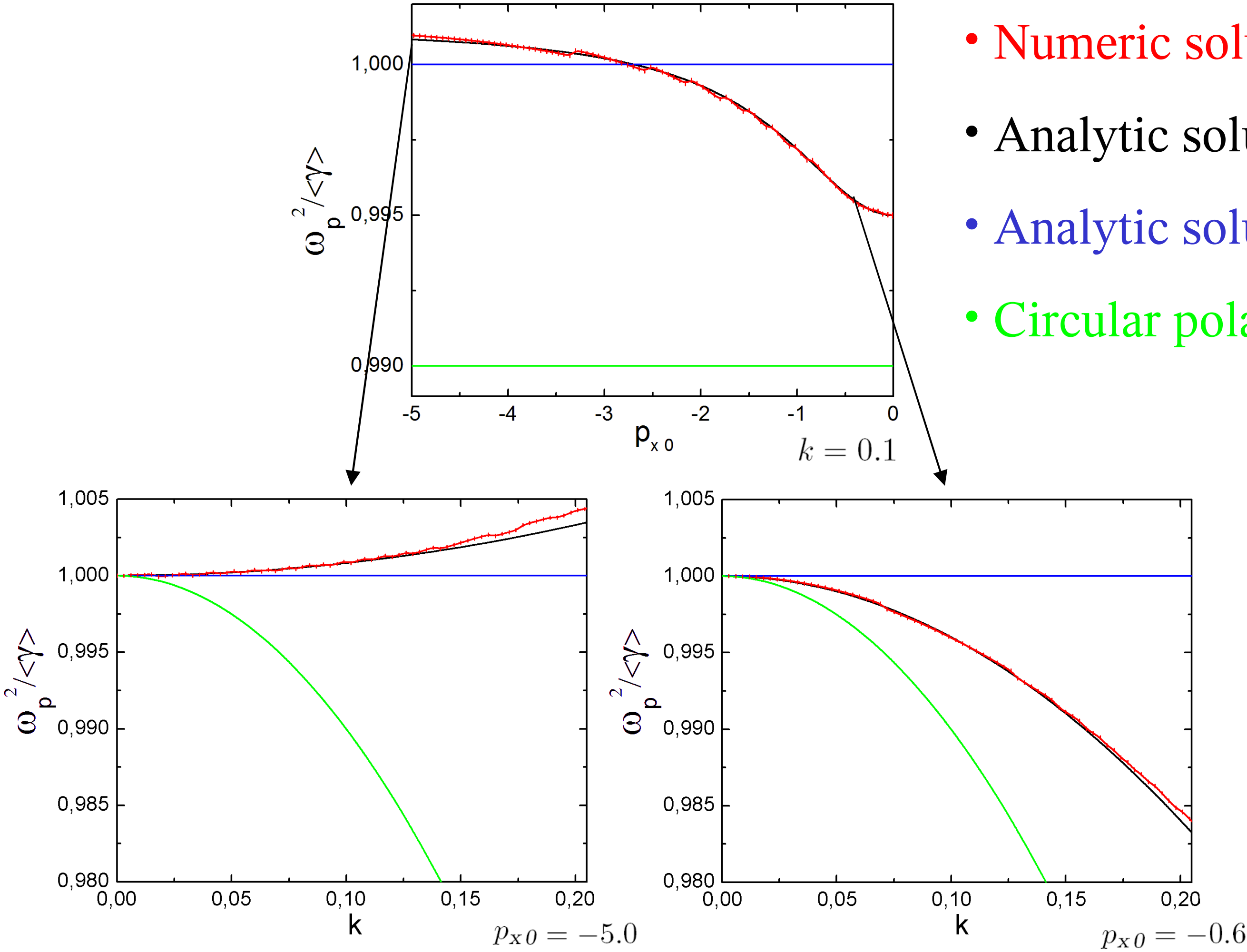
$$\frac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle} = \omega^2 - k^2$$

circular polarization

RWTH AACHEN UNIVERSITY

Runge-Kutta-Integration

Dispersion relation



- Numeric solution (exact)
- Analytic solution $\mathcal{O}(k^2)$
- Analytic solution $\mathcal{O}(k^0)$
- Circular polarization

RWTHAACHEN UNIVERSITY

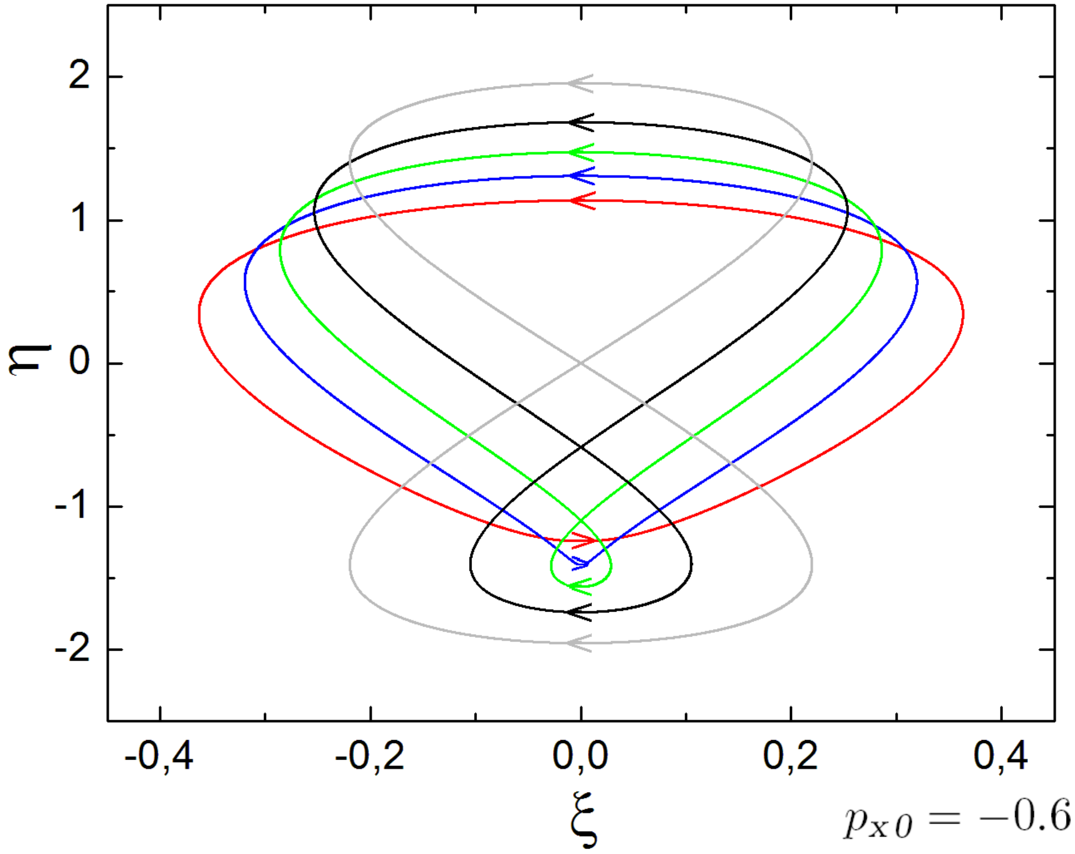
Intermediate plasma density

Transition between eight-like and circle-like trajectories.

Hard to handle analytically - no expansion possible.

$\rightarrow$ Numerical treatment (Runge-Kutta-Integration): k=0...1 ($\frac{\omega_{\mathrm{p}}^2}{\langle\gamma\rangle}$ =1...0)

Transition



- k=0.38 - distorted circle
- k=0.4
- k=0.41
- k=0.415
- k=0.42 - figure eight

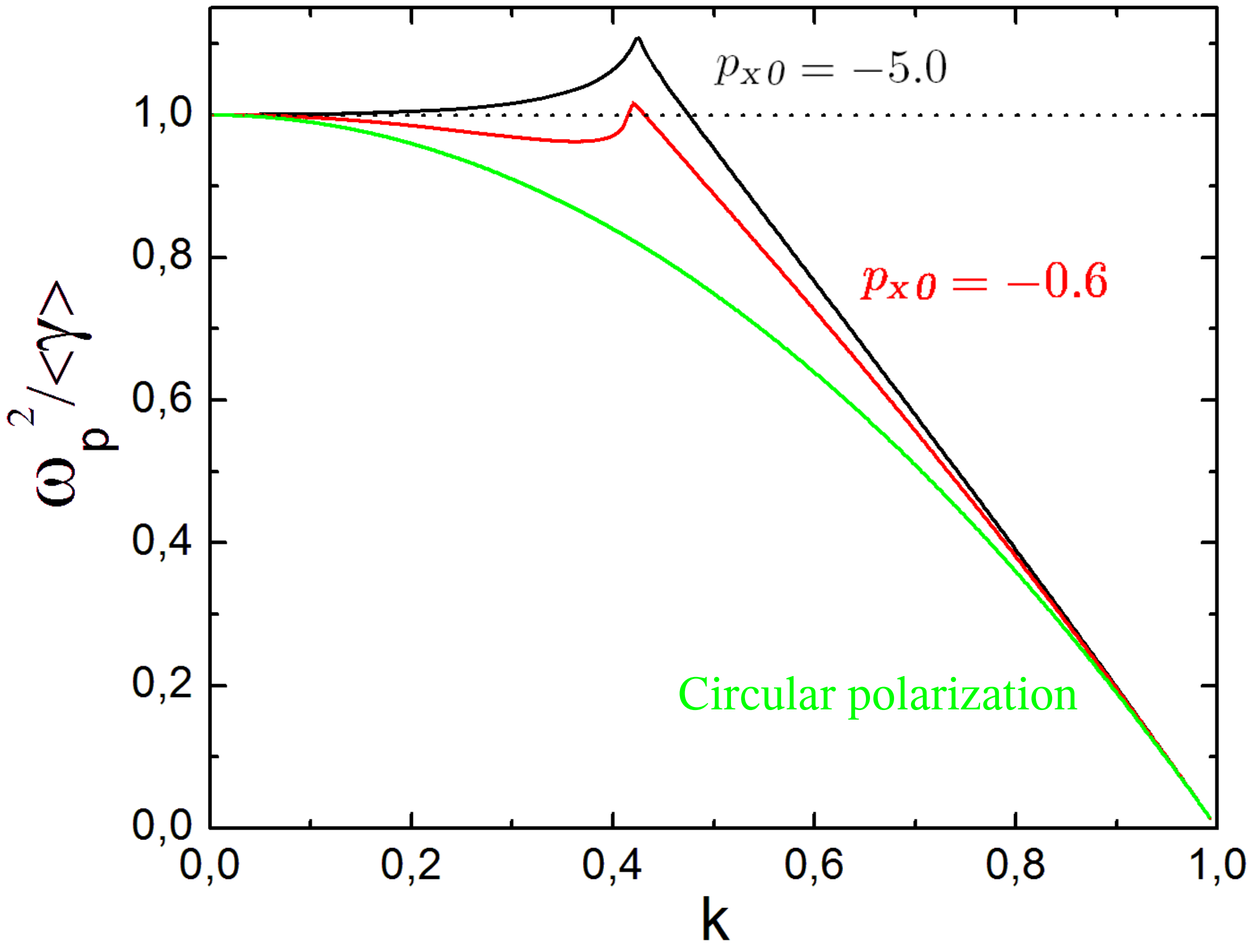
Summary

- Simple set of differential equations (in proper time).
- Periodic solutions.
- Effective plasma frequency is $\frac{\omega_{\mathrm{p}}^2}{\langle \gamma \rangle}$.
- Small plasma densities:
  Eight-like trajectories with higher harmonics -η odd, ξ even.
- Critical plasma density:
  Circle-like trajectories with shift and higher harmonics -η, ξ odd and even.
- Transition between circle-like and eight-like trajectories.
- Dispersion relation for linear polarization (density bunching) differs from that for circular polarization.

RWTH AACHEN UNIVERSITY

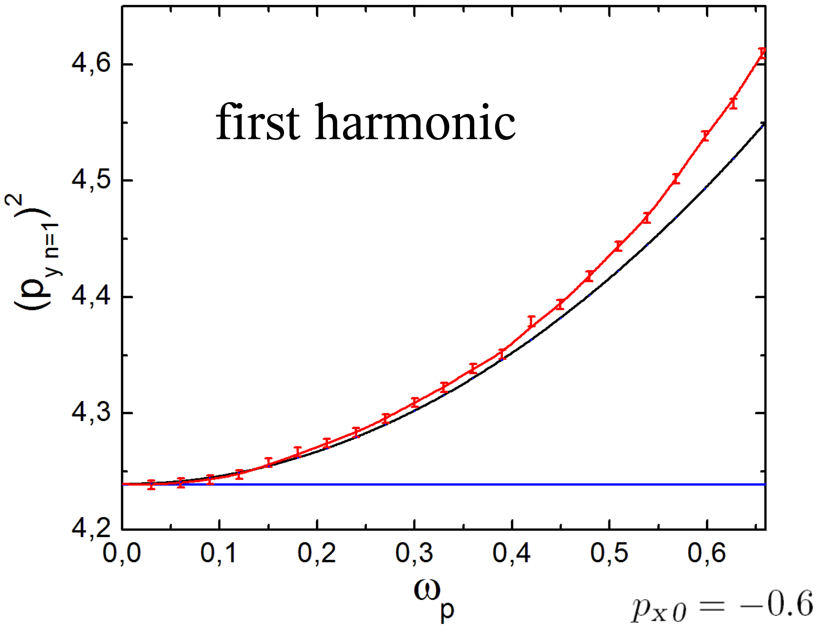
Transition

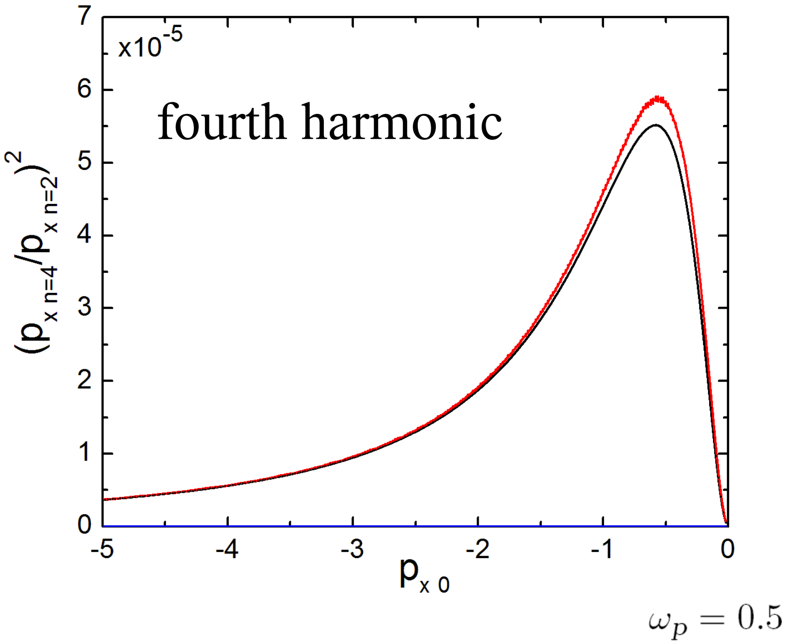
Dispersion relation



Runge-Kutta-Integration

Amplitudes

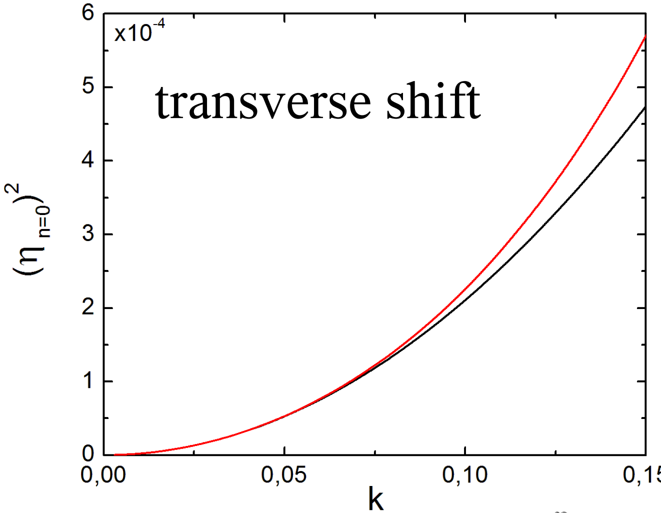




- Numeric solution (exact)
- Analytic solution $\mathcal{O}(\omega_{\mathrm{p}}^2)$
- Analytic solution $\mathcal{O}(\omega_{\mathrm{p}}^0)$ - Vacuum

RWTHAACHEN UNIVERSITY

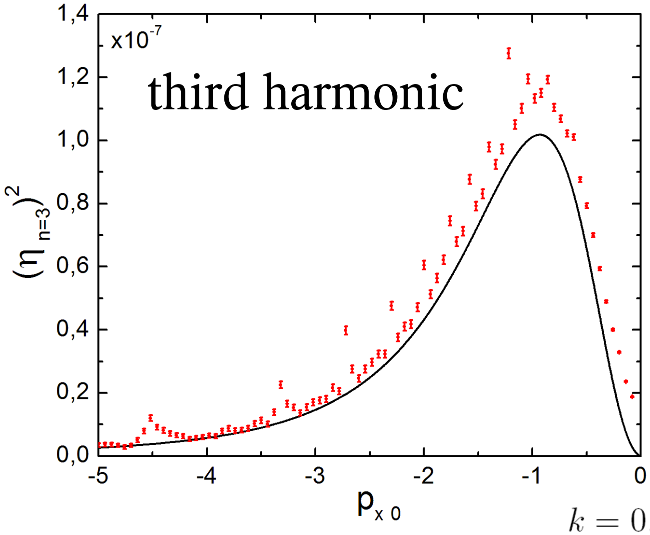
Runge-Kutta-Integration

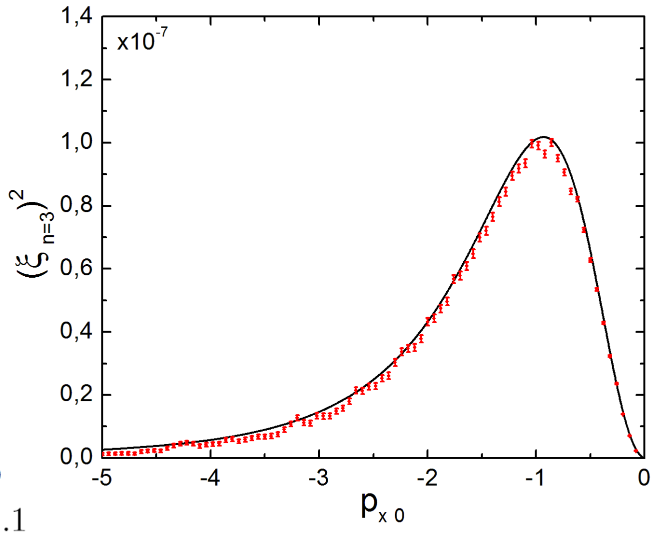


$p_{x\,0} = -0.6$

Amplitudes

- Numeric solution (exact)
- Analytic solution $\mathcal{O}(k^2)$

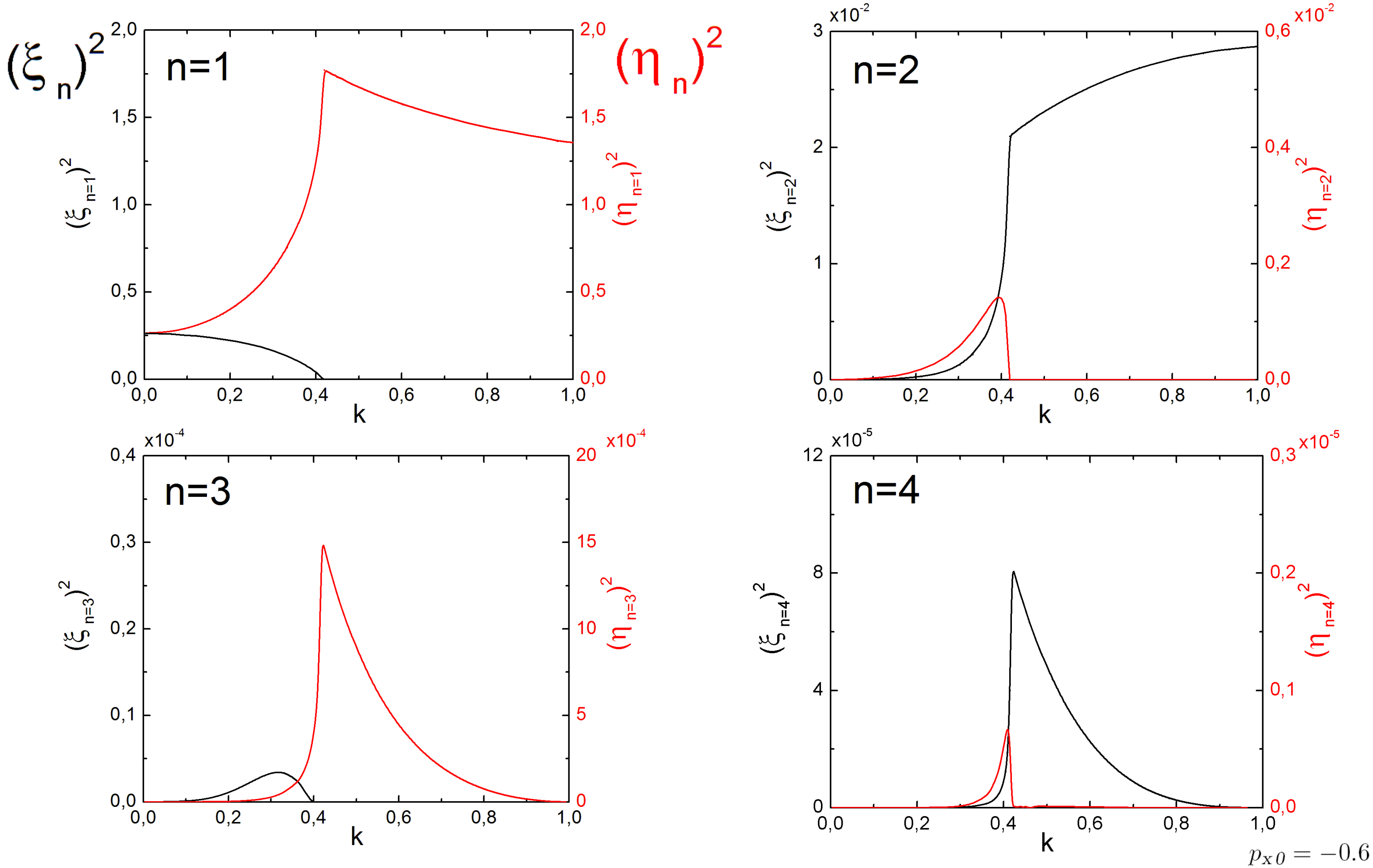




$k = 0.1$

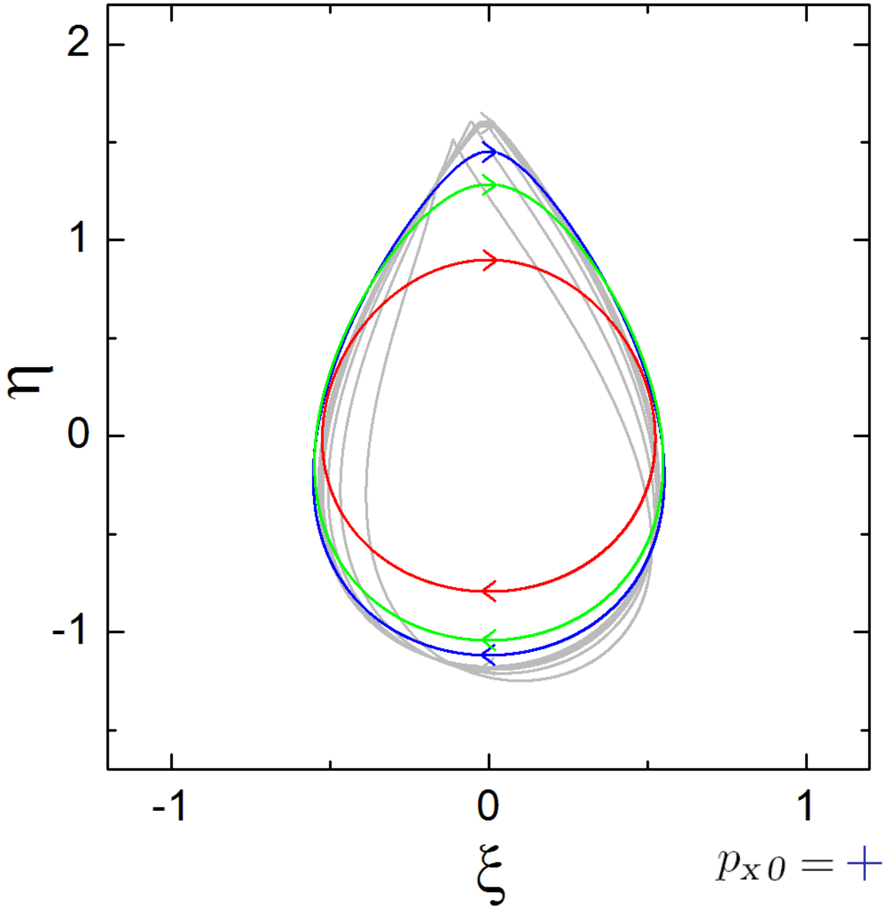
RWTH AACHEN UNIVERSITY

Transition



odd harmonics even harmonics

Transition



$p_{x\,0} = +0.6$

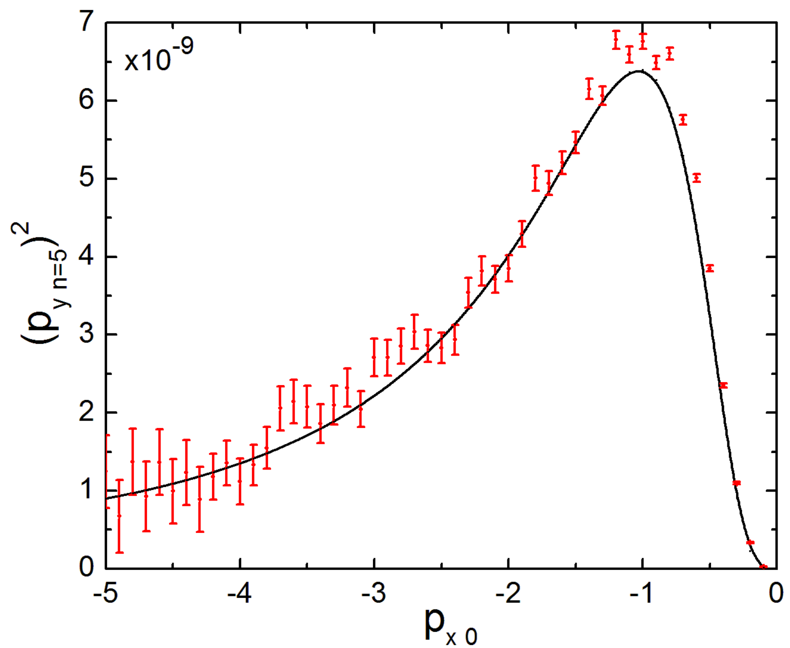
No transition

No periodic eight-like trajectory with $p_{x\,0} = +0.6$

Even higher Orders

$$\xi_{n=6} = -\frac{3}{16}\frac{\omega_\mathrm{p}^4}{\langle\gamma\rangle^2}\frac{1}{8^4}\hat{\eta}^6 \sin(6\phi_\tau)$$

$$\eta_{n=5} = \frac{23}{32}\frac{\omega_\mathrm{p}^4}{\langle\gamma\rangle^2}\frac{1}{8^4}\hat{\eta}^5 \cos(5\phi_\tau)$$



→ Scaling law:

$$\xi_{2n+2} \sim -\frac{1}{8}\hat{\eta}^2 \left(-\frac{\omega_\mathrm{p}^2}{\langle\gamma\rangle}\frac{1}{8^2}\hat{\eta}^2\right)^n \sin\left((2n+2)\phi_\tau\right)$$

$$\eta_{2n+1} \sim \hat{\eta} \left(-\frac{\omega_\mathrm{p}^2}{\langle\gamma\rangle}\frac{1}{8^2}\hat{\eta}^2\right)^n \cos\left((2n+1)\phi_\tau\right)$$

← Sprangle et al.

System of units

- Dimensionless expressions:

$$t' = \omega t; \quad \boldsymbol{x}' = \frac{\omega}{c}\boldsymbol{x}; \quad \boldsymbol{v}' = \frac{1}{c}\boldsymbol{v}; \quad \boldsymbol{p}' = \frac{1}{mc}\boldsymbol{p}$$

$$q' = \frac{1}{e}q; \quad n' = \frac{1}{n_0}n; \quad \boldsymbol{E}' = \frac{e}{m\omega c}\boldsymbol{E}; \quad \boldsymbol{B}' = \frac{e}{m\omega c}\boldsymbol{B}$$

- Phase: $\phi' = t' - \frac{1}{v'_{\text{ph}}} x'$ $v_{\text{ph}} = \frac{\omega}{k}$

- CGS-System