Linearly polarized waves with constant phase velocity in relativistic plasmas

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The propagation of plane linearly polarized electromagnetic waves in cold plasmas at relativistic intensities is studied analytically under the assumption of a constant phase velocity. A system of coupled relativistic harmonic oscillators for the Lagrangian coordinates of the particles is derived. Based on this model, a perturbation expansion is carried out to solve the equations for small plasma densities on the one hand and nearly critical plasma densities on the other hand. In both cases fully relativistic expressions for the particle trajectories and the dispersion relations are derived. For small plasma densities the particle trajectory approaches the vacuum figure-eight orbit. For plasma densities close to the critical density a deformed circular orbit is found that differs from the commonly considered almost-transverse solution. Finally, the transition between the two classes of solutions at intermediate plasma densities is numerically investigated. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2760209]

I. INTRODUCTION

For many applications the propagation of a laser pulse in a relativistic plasma is of great importance, including such different areas as higher-order harmonic generation,[1] formation of plasma channels[2] or particle acceleration.[3–5] The fundamental nonlinear relativistic interaction leads to a number of basic effects, e.g., self-focusing and self-modulation,[6,7] pulse compression and attosecond pulse generation,[8–11] filamentation,[12,13] and the appearance of strong localized fields and solitons.[14–16] Based on relativistic pulse propagation different electron acceleration mechanisms, like direct laser acceleration,[17] wake-field acceleration,[18] and laser bubble acceleration[19] are currently investigated. Accelerated electrons are also a source of collimated short wavelength radiation.[20,21] In all cases it is important to understand the underlying propagation properties.

In this work, we pursue an analytical treatment of the propagation properties of a linearly polarized plane electromagnetic wave in relativistic plasmas. It is our basic goal to derive dispersion relations for wave propagation that are valid in the whole regime from weak up to ultrarelativistic laser intensities. The present approach is based on a Lagrangian description of particle motion leading to a set of coupled relativistic oscillators for the transverse and longitudinal motions. Within this framework, particle trajectories and dispersion relations are derived systematically both for low and near-critical density plasmas. The relationship of the present results with previous work is discussed in detail. One major result is the demonstration of symmetry breaking of the orbit in the transition regime between low and near-critical densities. We first briefly review previous approaches to the subject.

Analytical investigations of the propagation of plane electromagnetic waves in cold plasmas customarily begin with the equations of Akhiezer and Polovin.[22,23] In this model the electrons are treated relativistically, the ions are assumed stationary, and thermal effects are neglected. The model depends on three main assumptions. First, the restriction on plane waves with constant phase velocity implies one-dimensional wave geometry and time-independent amplitudes. It is therefore assumed that the transverse size of the pulse is large compared to c/ω_p and that the pulse envelope changes only quasistatically. Second, the plasma is assumed cold. In the relativistic case this assumption is not very restrictive, since the thermal velocity can often be neglected against the directed quiver velocity. Third, the ion motion is not taken into account. This assumption is applicable if the ion velocity is negligible in comparison to the electron velocity, for example as a consequence of the larger inertial mass.

The equations of Akhiezer and Polovin follow from Maxwell equations and the relativistic equations of motion, which describe the coupling of the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ with the electron density n and the velocity $\boldsymbol{v}$,

$$-(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B})=\frac{d}{dt}(\gamma\boldsymbol{v}), \tag{1a}$$

$$\nabla\times\boldsymbol{B}-\partial_t\boldsymbol{E}=-\omega_p^2 n\boldsymbol{v}, \tag{1b}$$

$$\nabla\times\boldsymbol{E}+\partial_t\boldsymbol{B}=0, \tag{1c}$$

$$\nabla\cdot\boldsymbol{E}=\omega_p^2(1-n), \tag{1d}$$

$$\nabla\cdot\boldsymbol{B}=0, \tag{1e}$$

$$0=\partial_t n+\nabla\cdot(n\boldsymbol{v}). \tag{1f}$$

Here ω_p is the plasma frequency and $\gamma=1/\sqrt{1-\boldsymbol{v}^2}$ denotes the relativistic factor. Dimensionless quantities are used throughout the paper. They are connected with the dimensional quantities (dashed) by $t=\omega t'$, $x=x'\omega/c$, $p=p'/(mc)$,

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$E = eE'/(m\omega c)$, and $n = n'/n_0$, where n_0 is the density of the ions at rest. Within this framework the frequency of the wave is equal to unity, $\omega = 1$. When needed for clarity ω is reintroduced.

Under the assumption of plane waves with wavenumber k and a constant phase velocity $v_{\rm ph} = \omega/k$ propagating in the x-direction all quantities depend only on the phase $\phi = t - x/v_{\rm ph}$. This corresponds to a quasistatic approximation of the pulse. It allows one to transform the partial differential equations into ordinary differential equations.

In the case of linear polarization in the y-direction Eqs. (1) reduce to two nonlinear, coupled differential equations,[22]

$$p_y'' + \frac{\sigma^2 v_{\rm ph}^2}{v_{\rm ph}\gamma - p_x} p_y = 0, \tag{2a}$$

$$(v_{\rm ph}p_x - \gamma)'' + \frac{\omega_p^2 v_{\rm ph}^2}{v_{\rm ph}\gamma - p_x} p_x = 0, \tag{2b}$$

where $\sigma^2 = \omega_p^2 v_{\rm ph}/(v_{\rm ph}^2 - 1)$ is a constant, $\boldsymbol{p} = \gamma \boldsymbol{v}$ is the dimensionless relativistic momentum, and primes denote derivatives with respect to the phase. The electron density and all field components can be expressed in terms of the momenta p_y and p_x. With this approach Akhiezer and Polovin[22] investigated relativistic longitudinal oscillations and circularly polarized electromagnetic waves. The latter obey the famous relativistic dispersion relation

$$c^2k^2 = \omega^2 - \frac{\omega_p^2}{\gamma}.$$

Furthermore, they briefly studied linearly polarized waves, considering the lowest order vacuum solution as well as a critical density solution. In the weakly relativistic regime Lünow[24] derived in addition a solution up to the third-order in the amplitude. On the other hand Kaw and Dawson[23] and Max and Perkins[25] focused their attention to the almost-transverse solution at critical densities and derived a weakly relativistic as well as an ultrarelativistic solution. In later work ion motion was also included.[26]

Another form of Eq. (2) can be derived using the potentials $A_y = p_y$ and $\varphi = \gamma - v_{\rm ph}p_x$ instead of the momenta,[27]

$$A_y'' + \frac{\sigma^2 v_{\rm ph}^2}{\sqrt{\varphi^2 + (1 + A_y^2)(v_{\rm ph}^2 - 1)}} A_y = 0, \tag{3a}$$

$$\varphi'' + \frac{\sigma^2 v_{\rm ph}^2}{\sqrt{\varphi^2 + (1 + A_y^2)(v_{\rm ph}^2 - 1)}} \varphi = \sigma^2 v_{\rm ph}. \tag{3b}$$

Similar equations were used by Sprangle *et al.*[28–31] and later by Zhmoginov and Fraiman[32] to investigate the generation of higher harmonics in the transverse component for the case of small plasma densities. Borovsky *et al.*[27] also considered nonperiodic solutions.

A totally different approach was first introduced by Winkles *et al.*[33] and was later used extensively by Clemmow.[34] They considered a system $\tilde{S}$ moving with the velocity $1/v_{\rm ph}$ relative to the laboratory system. In $\tilde{S}$ the ions are streaming with $1/v_{\rm ph}$. All quantities are then space-independent and the magnetic field as well as the density modulation disappears. The only nonlinearity left is the relativistic factor $\gamma = \sqrt{1 + p_x^2 + p_y^2}$,[34]

$$\frac{d^2}{d\tilde{t}^2}\tilde{p}_y + \frac{\omega_p^2}{\tilde{\gamma}}\tilde{p}_y = 0, \tag{4a}$$

$$\frac{d^2}{d\tilde{t}^2}\tilde{p}_x + \frac{\omega_p^2}{\tilde{\gamma}}\tilde{p}_x = -\frac{\omega_p^2}{v_{\rm ph}}. \tag{4b}$$

All tilded expressions are connected with the laboratory frame expressions by a Lorentz transformation. In many cases the solution is simplified by this approach.[34–36] Many results were summarized by Decoster.[37] Nevertheless, this system is not convenient in the case of a vanishing plasma density since that implies $v_{\rm ph} \to 1$.

In this paper we use another equivalent system of equations convenient in the small density regime as well as in the critical density regime. Furthermore, we focus our attention on completely relativistic solutions. A perturbation expansion in the plasma density is carried out to generalize the vacuum solution. The fully consistent solution up to order ω_p^2 is calculated and the corresponding dispersion relation up to order ω_p^4 is derived. The differences to previous results are pointed out. For nearly critical plasma densities an expansion in the wavenumber $k = 1/v_{\rm ph}$ is carried out. The almost-transverse solution is briefly reviewed and a different type of solution is introduced. The expansion up to order k^2 yields the particle trajectories and the dispersion relation. Finally the transition between the large and the small density solutions at intermediate densities is discussed. All analytical results are validated against numerical solutions of the basic equations using a fourth-order Runge-Kutta method.

The paper is organized as follows: In Sec. II, the basic set of equations is introduced and some important relations are deduced. The small density regime is discussed in Sec. III and the critical density regime is investigated in Sec. IV. The behavior at intermediate plasma densities is finally considered in Sec. V.

II. THE EQUATIONS IN PROPER TIME

As presented in the Introduction, the form of the equations depends strongly on the choice of variables. Instead of the momenta as a function of the phase ϕ as in (2) we use Lagrangian coordinates as a function of the proper time τ,

$$\eta(\tau) = y - y_0, \tag{5a}$$

$$\xi(\tau) = x - x_0. \tag{5b}$$

Together with the substitution

$$\frac{d}{d\phi} = \frac{dt}{d\phi}\frac{d\tau}{dt}\frac{d}{d\tau} \to \frac{v_{\rm ph}}{v_{\rm ph}\gamma - p_x}\frac{d}{d\tau}, \tag{6}$$

which follows from $\phi = t - x/v_{\rm ph}$ and $dt/d\tau = \gamma$, the equations of Akhiezer and Polovin can be transformed into a simpler set of equations:

$$\ddot{\eta} + \sigma^2 v_{\rm ph}\gamma\eta = \sigma^2 \eta p_x, \tag{7a}$$

$$\ddot{\xi}+\omega_p^2\gamma\xi=-\sigma^2\eta p_y, \tag{7b}$$

where the dots denote derivatives with respect to the proper time, and $p_y=\dot{\eta}$ and $p_x=\dot{\xi}$ are the momenta.

This set of differential equations corresponds to a system of coupled relativistic harmonic oscillators and the fundamental symmetry between η and ξ is clarified. It describes the full relativistic coupling between the transverse electromagnetic and the longitudinal electrostatic oscillations. These equations for the Lagrange coordinates η and ξ serve as the basic equations for the subsequent analysis. The equations are nonlinear and have to be solved self-consistently. Because in (7a) and (7b) no variables appear in the denominator, all terms can easily be expanded, especially if the solutions are expressed in terms of harmonic functions.

The field components and the electron density are calculated from the Lagrangian coordinates in the form

$$E_x=\omega_p^2\xi, \tag{8a}$$

$$B_z=\sigma^2\eta, \tag{8b}$$

$$E_y=v_{\mathrm{ph}}B_z, \tag{8c}$$

$$n=\frac{v_{\mathrm{ph}}}{v_{\mathrm{ph}}-v_x}. \tag{8d}$$

From (7a) and (7b) follows immediately the conservation of energy,

$$\gamma=\gamma|_{\tau=0}+\frac{\sigma^2 v_{\mathrm{ph}}}{2}(\eta_{\tau=0}^2-\eta^2)+\frac{\omega_p^2}{2}(\xi_{\tau=0}^2-\xi^2). \tag{9}$$

Similar equations were derived by Lünow[24] but used only to investigate the one-dimensional plasma oscillation ($\eta=0$). In this paper we are mainly interested in linearly polarized waves with two degrees of freedom ($\eta\neq 0$ and $\xi\neq 0$).

Following previous work, we restrict our attention to periodic solutions with closed particle trajectories. Therefore, one initial condition has to be adjusted properly. The resulting solutions are the generalizations of the closed eight-like vacuum solution in a plasma. A coupling with additional electrostatic oscillations, leading to nonclosed orbits, was for example, investigated by Borovsky *et al.*[27] and will be considered elsewhere.

The mathematical condition for periodic solutions is $\langle \boldsymbol{v}\rangle_t=0$, where the time average is performed in the inertial frame in which the ions are at rest. This is equivalent to the condition $\langle \boldsymbol{p}\rangle=0$, where the average is now performed over a period of the proper time. For numerical solutions corresponding initial conditions can be found iteratively. For periodic solutions the frequency in the laboratory frame ω and the frequency in the rest frame of the electron Ω are related by

$$\Omega=\left\langle\frac{d\phi}{d\tau}\right\rangle=\left\langle\frac{1}{v_{\mathrm{ph}}}(v_{\mathrm{ph}}\gamma-p_x)\right\rangle=\langle\gamma\rangle\omega, \tag{10}$$

where $\langle\gamma\rangle$ is the mean energy, averaged over the proper time, and ω is reintroduced for clarity. The corresponding phase $\phi_\tau=\Omega\tau$ is the phase in the rest frame of the electron and is due to the averaging process not equal to the laboratory phase ϕ.

Because in (7a) and (7b) the derivatives are with respect to the proper time, all the calculated solutions depend on the phase ϕ_τ. This is convenient since it allows one a simple conversion between the coordinates and the momenta. For each result, the transformation from the argument ϕ_τ to ϕ can be performed.

III. SMALL DENSITY REGIME

In this section we investigate the problem for small plasma densities. A perturbation expansion in ω_p is carried out to solve Eqs. (7a) and (7b) self-consistently up to order ω_p^2. This yields a dispersion relation up to order ω_p^4.

For convenience we choose the initial condition

$$\xi(0)=0,\quad \eta(0)\neq 0, \qquad p_x(0)\neq 0,\quad p_y(0)=0. \tag{11}$$

In the lowest order, i.e. order ω_p^0, the solution corresponds to the vacuum solution of a single particle in an electromagnetic wave.[22,38] The trajectory is the famous figure-eight and consists of a transverse component η and a longitudinal component ξ, which oscillates with twice the transverse frequency,

$$\eta=\hat{\eta}\cos(\phi_\tau), \tag{12a}$$

$$\xi=-\tfrac{1}{8}\hat{\eta}^2\sin(2\phi_\tau). \tag{12b}$$

This vacuum solution already leads to the dispersion relation of order ω_p^2,[22]

$$v_{\mathrm{ph}}=1+\frac{1}{2}\frac{\omega_p^2}{\langle\gamma\rangle}. \tag{13}$$

To evaluate Eqs. (7a) and (7b) up to order ω_p^2 we use the identity

$$v_{\mathrm{ph}}\gamma-p_x=(v_{\mathrm{ph}}\gamma-p_x)|_{\tau=0}+\omega_p^2\frac{v_{\mathrm{ph}}}{2}(\eta_{\tau=0}^2-\eta^2)+\omega_p^2\int_0^\tau(\gamma-v_{\mathrm{ph}}p_x)\xi d\tau, \tag{14}$$

which follows from the second differential equation (7b) together with the conservation of energy (9). This identity indicates that $v_{\mathrm{ph}}\gamma-p_x$ of order ω_p^2 can be calculated from the vacuum solution.

Therefore, the first differential equation (7a),

$$\ddot{\eta}+\sigma^2(v_{\mathrm{ph}}\gamma-p_x)\eta=0, \tag{15}$$

can be solved for η independent of the second differential equation.

In the next step we use this solution for η together with the expression for the energy (9) in the second differential equation,

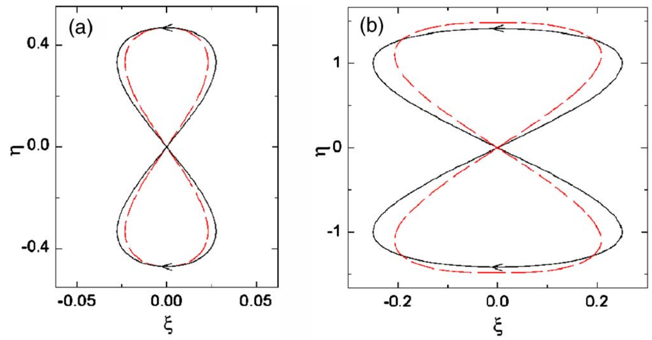

FIG. 1. (Color online) The particle trajectory up to order ω_p^0 (solid) and ω_p^2 (dashed) in (a) a weakly relativistic $\langle\gamma\rangle=1.06$ and (b) ultrarelativistic case $\langle\gamma\rangle=10^3$. The propagation direction is the ξ-direction. For better visual convenience, the results are compared at a relatively high density.

$$\ddot{\xi}+\omega_p^2\gamma\xi=-\sigma^2\eta p_y,$$

and also obtain ξ. This approach yields the solution fully consistent up to order ω_p^2,

$$\eta=\hat{\eta}\left(1+\frac{\omega_p^2}{\langle\gamma\rangle}\frac{1}{64}\hat{\eta}^2\right)\cos(\phi_\tau)-\frac{\omega_p^2}{\langle\gamma\rangle}\frac{3}{256}\hat{\eta}^3\cos(3\phi_\tau)+\mathcal{O}(\omega_p^4), \tag{16a}$$

$$\xi=-\frac{1}{8}\hat{\eta}^2\left(1-\frac{\omega_p^2}{\langle\gamma\rangle}\frac{1}{4}\left(1-\frac{13}{32}\hat{\eta}^2\right)\right)\sin(2\phi_\tau)+\frac{\omega_p^2}{\langle\gamma\rangle}\frac{5}{256}\frac{1}{8}\hat{\eta}^4\sin(4\phi_\tau)+\mathcal{O}(\omega_p^4), \tag{16b}$$

where $\phi_\tau=\Omega\tau$ with $\Omega=\langle\gamma\rangle$ is the phase in the rest frame of the electron and $\hat{\eta}$ is a parameter, that corresponds to the amplitude in the vacuum solution (12a). For a nonvanishing plasma density it is defined by the mean energy, $\langle\gamma\rangle=1/\sqrt{1-\hat{\eta}^2/2}$. Momentum expressions follow from a derivative of (16a) and (16b) with respect to τ.

As expected from Lorentz invariance, the effective expansion parameter is $\omega_p^2/\langle\gamma\rangle$ instead of ω_p^2. For $\omega_p\neq 0$ the amplitude of the transverse component η and the longitudinal component ξ receive corrections of order $\omega_p^2/\langle\gamma\rangle$. Furthermore, both components include higher harmonics: odd harmonics in the transverse direction and even harmonics in the longitudinal direction. The corrections to the free particle solution are illustrated in Fig. 1.

To compare parts of the result with previous work and for the sake of completeness we express the solution (16) in terms of the laboratory phase. For this purpose, we integrate the identity

$$\frac{d\phi}{d\tau}=\frac{1}{v_{\text{ph}}}(v_{\text{ph}}\gamma-p_x) \tag{17}$$

over τ using

$$v_{\text{ph}}\gamma-p_x=\langle\gamma\rangle\left(1+\frac{\omega_p^2}{\langle\gamma\rangle}\left(\frac{1}{2}-\frac{3}{16}\hat{\eta}^2\cos(2\phi_\tau)\right)\right)+\mathcal{O}(\omega_p^4),$$

which follows from the solution of order ω_p^2. This leads to the transformation equation,

$$\phi=\phi_\tau-\frac{\omega_p^2}{\langle\gamma\rangle}\frac{3}{32}\hat{\eta}^2\sin(2\phi_\tau)+\mathcal{O}(\omega_p^4). \tag{18}$$

The solution of order ω_p^2, expressed in terms of the momenta and as functions of the laboratory phase ϕ is therefore

$$p_y=-\langle\gamma\rangle\hat{\eta}\left(1+\frac{\omega_p^2}{\langle\gamma\rangle}\frac{1}{16}\hat{\eta}^2\right)\sin(\phi)-\omega_p^2\frac{3}{256}\hat{\eta}^3\sin(3\phi)+\mathcal{O}(\omega_p^4), \tag{19a}$$

$$p_x=-\frac{1}{4}\langle\gamma\rangle\hat{\eta}^2\left(1-\frac{\omega_p^2}{\langle\gamma\rangle}\frac{1}{4}\left(1-\frac{13}{32}\hat{\eta}^2\right)\right)\cos(2\phi)-\omega_p^2\frac{7}{512}\hat{\eta}^4\cos(4\phi)+\omega_p^2\frac{3}{128}\hat{\eta}^4+\mathcal{O}(\omega_p^4). \tag{19b}$$

It is important to note that the constant term in p_x as a function of ϕ is no contradiction to the periodicity condition, $\langle\boldsymbol{p}\rangle=0$, since the average has to be performed over the proper time. The third harmonic in the transverse component, $p_{y3}=-\omega_p^2 3/256\hat{\eta}^3\sin(3\phi)$, coincides exactly with the result of Sprangle[30] and Mori,[31] derived in the framework of harmonic generation in underdense plasma using the wave equations for the potentials. However, the longitudinal component and the amplitude corrections of the fundamental harmonics were not investigated. The result for the third harmonic coincides also with the later result from Zhmoginov and Fraiman.[32]

For a comparison with these results it is most suitable to express (19a) in terms of the transverse momentum or the vector potential parameter $\hat{a}$, which obeys the relation

$$\hat{\eta}=\frac{\hat{a}}{\langle\gamma\rangle}. \tag{20}$$

This parameter is linked with the mean energy by $\langle\gamma\rangle=\sqrt{1+\hat{a}^2/2}$.

Together with the trajectories of order ω_p^2 (16) we also obtain σ^2 of order ω_p^2. From this we can already derive the phase velocity up to order ω_p^4. For this purpose we solve the expression of $\sigma^2=\omega_p^2 v_{\text{ph}}/(v_{\text{ph}}^2-1)$ for v_{ph}:

$$v_{\text{ph}}=1+\frac{1}{2}\frac{\omega_p^2}{\langle\gamma\rangle}+\frac{3}{8}\left(1-\frac{1}{8}\hat{\eta}^2\right)\frac{\omega_p^4}{\langle\gamma\rangle^2}. \tag{21}$$

This result is fully consistent up to the order considered. It extends the result of Rax and Fisch,[39] who also accounted for some higher-order corrections in the solution, but neglected a term of order ω_p^4 in the expansion of v_{ph}. Furthermore, they calculated $\sqrt{2-1/v_{\text{ph}}^2}$ instead of v_{ph} due to a different use of the phase.

For the stationary solutions (19) the phase velocity of the higher harmonics is equal to the one of the fundamental. This can be verified by evolution equations similar to Eqs. (11a) and (11b) in Ref. 39 including the neglected terms,

$$9\left(1-\frac{1}{v_{\text{ph }3}^2}\right)=\frac{\omega_p^2}{\langle\gamma\rangle}-\frac{\omega_p^4}{\langle\gamma\rangle}\frac{3}{32}\frac{\hat{\eta}^3}{\hat{a}_3}\cos\left(\frac{3}{v_{\text{ph }3}}-\frac{3}{v_{\text{ph}}}\right),$$

$$6\frac{d\hat{a}_3}{dt}=\frac{\omega_p^4}{\langle\gamma\rangle}\frac{3}{32}\hat{\eta}^3\sin\left(\frac{3}{v_{\text{ph }3}}-\frac{3}{v_{\text{ph}}}\right).$$

These equations describe the time dependency of the transverse third harmonic amplitude $\hat{a}_3$ as a function of the difference between the phase velocity of the third harmonic $v_{\text{ph }3}$ and the fundamental phase velocity v_{ph}.[39] A stationary solution is $v_{\text{ph }3}=v_{\text{ph}}$ together with $\hat{a}_3=-\omega_p^2 3/256\hat{\eta}^3$, just as expected from (9a).

The expression for the phase velocity (21) is equivalent to the dispersion relation

$$c^2k^2=\omega^2-\frac{\omega_p^2}{\langle\gamma\rangle}+\frac{3}{32}\hat{\eta}^2\frac{\omega_p^4}{\langle\gamma\rangle^2\omega^2}+\mathcal{O}(\omega_p^6), \tag{22}$$

where ω is again reintroduced for clarity.

The comparison of (22) with the well-known exact dispersion relation for circular polarization ($\gamma=\langle\gamma\rangle$),

$$c^2k^2=\omega^2-\frac{\omega_p^2}{\langle\gamma\rangle}, \tag{23}$$

shows a crucial difference in the lowest nontrivial order.

The approach (14) used to calculate the solution of order ω_p^2 from the solution of order ω_p^0 can also be used to compute successively higher-order solutions. However, the calculation gets more and more cumbersome and leads to no further insight. Therefore, we restrict ourself to mention the result for the next higher harmonics,

$$p_{y\ 5}=-\frac{115}{32}\frac{\omega_p^4}{\langle\gamma\rangle}\frac{1}{8^4}\hat{\eta}^5\sin(5\phi_\tau)+\mathcal{O}(\omega_p^6), \tag{24a}$$

$$p_{x\ 6}=-\frac{9}{8}\frac{\omega_p^4}{\langle\gamma\rangle}\frac{1}{8^4}\hat{\eta}^6\cos(6\phi_\tau)+\mathcal{O}(\omega_p^6). \tag{24b}$$

To express this result in terms of the laboratory phase ϕ or to calculate the next order of the dispersion relation requires the full order ω_p^4 solution, including amplitude corrections of the fundamental and first higher harmonics. The calculation for $p_{y\ 5}$ yields

$$p_{y\ 5}=-\frac{85}{32}\frac{\omega_p^4}{\langle\gamma\rangle}\frac{1}{8^4}\hat{\eta}^5\sin(5\phi)+\mathcal{O}(\omega_p^6). \tag{25}$$

The results (24a) and (25) are not comparable to a similar derivation of Sprangle *et al.*[30] because they considered a different procedure, using a large transverse driver, which decouples the calculation of φ and A_y. That allowed them to calculate first all higher harmonics in φ and then separately all higher harmonics in A_y. As already mentioned by Zhmoginov and Fraiman[32] this separation is generally not possible. The difficulty does not arise for the lowest order higher harmonics (19). The result (25) differs also by a factor of order 1 from the result in Ref. 32 obtained by different approximations.

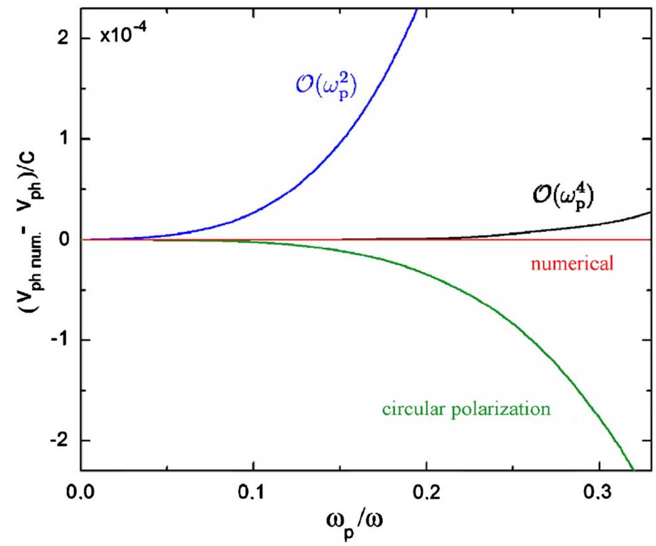

FIG. 2. (Color online) The difference between the numerically obtained phase velocity $v_{\text{ph num}}$ and the analytical results up to order ω_p^2 and ω_p^4, also compared with the phase velocity for the circularly polarized case, all evaluated for $p_x(0)=-0.6$.

Of course the general scaling law for higher harmonics stated in Ref. 30 is unchanged. An equivalent formulation expressed through ϕ_τ is

$$p_{y\ 2n+1}\sim-\langle\gamma\rangle\hat{\eta}\left(-\frac{\omega_p^2}{\langle\gamma\rangle}\frac{1}{8^2}\hat{\eta}^2\right)^n\sin((2n+1)\phi_\tau)+\mathcal{O}(\omega_p^{2n+2}), \tag{26a}$$

$$p_{x\ 2n+2}\sim-\frac{1}{4}\langle\gamma\rangle\hat{\eta}^2\left(-\frac{\omega_p^2}{\langle\gamma\rangle}\frac{1}{8^2}\hat{\eta}^2\right)^n\cos((2n+2)\phi_\tau)+\mathcal{O}(\omega_p^{2n+2}). \tag{26b}$$

Because of the exponential scaling higher harmonic generation in a collisionless, homogenous low density plasma is rather inefficient.

For small plasma densities all perturbation results agree excellently with the corresponding numerical solutions of the differential equations (7a) and (7b). For example, Fig. 2 illustrates the difference between the numerical and analytical results of the phase velocity and Fig. 3 compares the results of the higher harmonic $p_{y\ 5}$. The other quantities show a similar coincidence.

IV. CRITICAL DENSITY REGIME

Another interesting regime of the plasma density is the underdense region near the critical density. In this work we define the critical density as the highest density up to which the electromagnetic wave can propagate, including relativistic effects $\omega_p^2/\langle\gamma\rangle=\omega^2$.

At the critical density the wavenumber k of an electromagnetic wave or equivalently the index of refraction approaches zero, $k=1/v_{\text{ph}}\rightarrow 0$. Therefore, it is convenient to carry out a perturbation expansion in k to solve Eqs. (7a) and (7b) for nearly critical plasma densities.

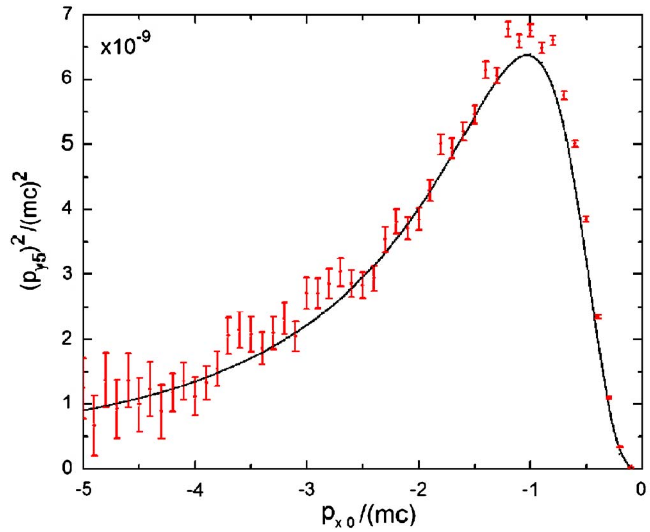

FIG. 3. (Color online) Numerical (dots) and analytical (solid) results for the square of the higher harmonic amplitude $(p_{y\,5})^2$ compared for $\omega_p^2=0.25$.

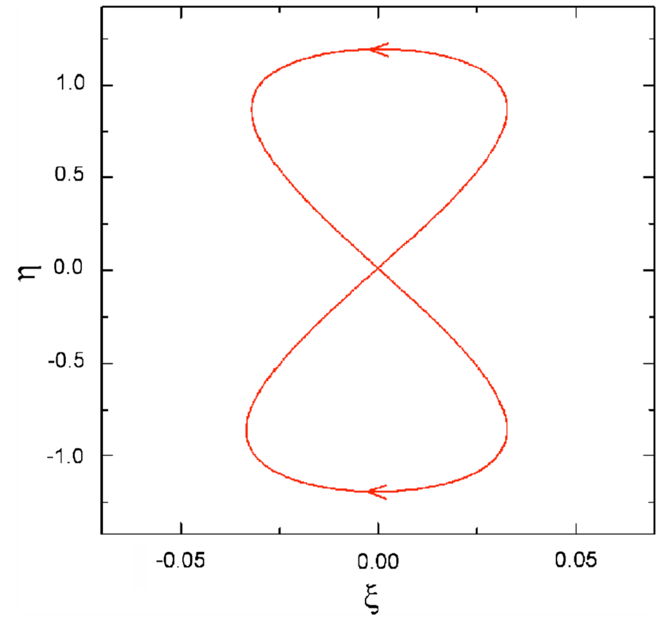

FIG. 4. (Color online) The numerically calculated trajectory at $k=0.1$ for an intermediate amplitude ($\langle\gamma\rangle=1.7$, i.e. $\hat{\eta}=1.14$).

Since $\sigma^2=\omega_p^2 k/(1-k^2)$ we can write (7a) and (7b) in the form

$$\ddot{\eta}+\frac{\omega_p^2}{1-k^2}\gamma\eta=\frac{\omega_p^2 k}{1-k^2}\eta p_x, \tag{27a}$$

$$\ddot{\xi}+\omega_p^2\gamma\xi=-\frac{\omega_p^2 k}{1-k^2}\eta p_y. \tag{27b}$$

One solution of these equations for small k is the well-known almost-transverse solution,[22,23,25,35] which we will review briefly. This solution follows under the assumption that the transverse part is large compared to the longitudinal part, $p_y \gg p_x$. The longitudinal wave is then treated as a driven wave. For $k=0$ the wave is electrostatic and purely transverse.[22] In this case Eq. (27a) corresponds to the famous nonlinear oscillator equation, usually considered to describe one-dimensional relativistic plasma oscillations. For a nonvanishing k the coupling term on the right-hand side of Eq. (27b) leads to a small longitudinal component.

The weakly relativistic limit of this case was investigated by Kaw and Dawson.[23] An equivalent solution in terms of the Lagrangian coordinates as a function of the proper time can be calculated performing an expansion of (27a) and (27b) in both k and $\hat{\eta}$:

$$\eta=\hat{\eta}\cos(\phi_\tau)-\tfrac{3}{64}\hat{\eta}^3\cos(3\phi_\tau)+\mathcal{O}(\hat{\eta}^4)+\mathcal{O}(k^2), \tag{28a}$$

$$\xi=-\tfrac{1}{6}k\hat{\eta}^2\sin(2\phi_\tau)+\mathcal{O}(\hat{\eta}^4)+\mathcal{O}(k^2), \tag{28b}$$

with the dispersion relation

$$\left(1-\frac{3}{8}\hat{\eta}^2\right)\omega_p^2\approx\omega^2-c^2k^2, \tag{29}$$

and $\langle\gamma\rangle=1/\sqrt{1-\hat{\eta}^2/2}=1+\hat{\eta}^2/4+\mathcal{O}(\hat{\eta}^4)$.

The ultrarelativistic limit, on the other hand, was investigated by Max and Perkins[25] with the parabolic approximation for p_y. This leads to the dispersion relation

$$\frac{\pi^2}{24}\frac{\omega_p^2}{\langle\gamma\rangle}\approx\omega^2-c^2k^2, \tag{30}$$

valid for $\langle\gamma\rangle$ sufficiently large. The harmonic content of p_y for this case was later studied by Mori *et al.*[31] Again the longitudinal part p_x oscillates with twice the frequency of p_y.

This fact remains even true for an arbitrary amplitude, although an analysis is then rather cumbersome and does not yield an explicit solution.[35] Nevertheless, it can be summarized that the almost-transverse solution is in general eight-like (Fig. 4).

In addition to the almost-transverse solution there exist other periodic, noneight-like solutions if the assumption $p_y \gg p_x$ is dropped.[40] One interesting question is how these solutions approach the only remaining eight-like vacuum solution for small plasma densities. This transition behavior is only straightforward for the eight-like almost-transverse solution.

At least one of the other solutions can be analytically described. The solution is circle-like and was already briefly mentioned by Akhiezer and Polovin.[22] It was later also considered by Clemmow[36] in the investigation of magnetized plasmas.

At the critical density, $k=0$, the new solution of Eqs. (27a) and (27b) is

$$\eta=\pm\hat{\eta}\cos(\phi_\tau)+\mathcal{O}(k), \tag{31a}$$

$$\xi=\hat{\eta}\sin(\phi_\tau)+\mathcal{O}(k). \tag{31b}$$

This solution is electrostatic ($\boldsymbol{B}=0$) and describes a circular trajectory in a plane spanned by the polarization- and propagation-directions. The coupling results, as in the case of circular polarization, from the relativistic gamma factor.

As pointed out by Clemmow[41] the coupling appears only in the relativistic case. In the nonrelativistic limit the solution

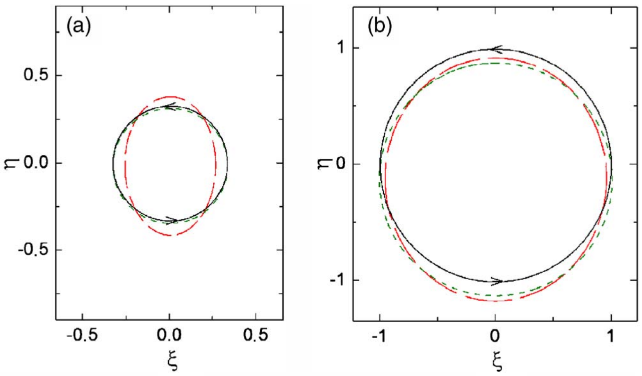

FIG. 5. (Color online) The particle trajectory up to order k^0 (solid), k^1 (short dashed), and k^2 (long dashed) in (a) a weakly relativistic $\langle\gamma\rangle=1.06$ and (b) ultrarelativistic case $\langle\gamma\rangle=10^3$ for $p_x(0)<0$. The propagation direction is the ξ-direction. Again k is chosen sufficiently large to visualize the corrections. The constant shift in the negative η-direction results from the Lorentz force.

corresponds just to two phase-matched electrostatic oscillations. But in contrast to the almost-transverse solution the expressions are formally valid for arbitrary amplitudes.

Below the critical density the solution is modified by the coupling on the right-hand side of (27a) and (27b) and has an electromagnetic part. To account for this, we perform a self-consistent expansion up to order k^2. An expansion up to order k is not sufficient to obtain a nontrivial dispersion relation.

The expansion is performed using the ansatz

$$\eta^{[2]} = \eta^{(0)} + k\eta^{(1)} + k^2\eta^{(2)} + \mathcal{O}(k^3) \tag{32}$$

for η and similar for ξ, γ, and ω_p^2. Together with the conservation of energy (9) and the consistency condition (10) we solve Eqs. (27a) and (27b) order by order to finally obtain

$$\eta = \pm\left\{\left(\hat\eta + k^2\frac{3(\hat\eta^4 - 33\hat\eta^2 + 48)}{16\hat\eta(3-\hat\eta^2)^2}\right)\cos(\phi_\tau) - k\frac{\hat\eta^2}{2(3-\hat\eta^2)}\cos(2\phi_\tau) - k^2\frac{9\hat\eta(1-\hat\eta^2)}{16(3-\hat\eta^2)^2}\cos(3\phi_\tau) + k\left(\frac{3\hat\eta^2}{2(3-\hat\eta^2)}\right)\right\} + \mathcal{O}(k^3), \tag{33a}$$

$$\xi = \left(\hat\eta + k^2\frac{8\hat\eta^6 - 19\hat\eta^4 + 99\hat\eta^2 - 144}{16\hat\eta(3-\hat\eta^2)^2}\right)\sin(\phi_\tau) - k\frac{\hat\eta^2}{2(3-\hat\eta^2)}\sin(2\phi_\tau) - k^2\frac{9\hat\eta(1-\hat\eta^2)}{16(3-\hat\eta^2)^2}\sin(3\phi_\tau) + \mathcal{O}(k^3), \tag{33b}$$

where this time $\hat\eta$ depends on the mean energy through $\langle\gamma\rangle=1/\sqrt{1-\hat\eta^2}$. This solution is illustrated in Fig. 5. In contrast to the almost-transverse solution, both η and ξ now contain even and odd higher harmonics. Furthermore, the corresponding amplitudes are equal in the lowest nonvanishing order. Another effect is the occurrence of a constant shift in the transverse direction, which is of order k. This shift can be easily explained by the Lorentz force, since $k\neq 0$ leads to $\boldsymbol{B}\neq 0$ $(B_z=\sigma^2\eta)$.

To express the solution in terms of the laboratory phase ϕ we use Eq. (17) once more. For simplicity only the lowest order corrections are transformed:

$$\frac{1}{v_{\rm ph}}(v_{\rm ph}\gamma - p_x) = \gamma - kp_x = \langle\gamma\rangle\left(1 - k\frac{3\hat\eta}{3-\hat\eta^2}\cos(\phi_\tau)\right) + \mathcal{O}(k^2).$$

This leads to the transformation equation,

$$\phi = \phi_\tau - k\frac{3\hat\eta}{3-\hat\eta^2}\sin(\phi_\tau) + \mathcal{O}(k^2), \tag{34}$$

and finally to the expressions for the momenta in terms of the laboratory phase:

$$p_y = \mp\left\{\langle\gamma\rangle\hat\eta\sin(\phi) + k\langle\gamma\rangle\frac{\hat\eta^2}{2(3-\hat\eta^2)}\sin(2\phi)\right\} + \mathcal{O}(k^2), \tag{35a}$$

$$p_x = \langle\gamma\rangle\hat\eta\cos(\phi) + k\langle\gamma\rangle\frac{\hat\eta^2}{2(3-\hat\eta^2)}\cos(2\phi) - k\langle\gamma\rangle\frac{3\hat\eta^2}{2(3-\hat\eta^2)} + \mathcal{O}(k^2). \tag{35b}$$

The results coincide well with a numerical integration of the equations of Akhiezer and Polovin (2). Thereby it is important to note that γ averaged over ϕ is not equal to $\langle\gamma\rangle$. A comparison with the results of Clemmow, however, is difficult, because he did not calculate expressions in the laboratory frame, in which the ions are at rest and in which experiments are usually performed. Furthermore, he used a very different notation, initially needed to account for the external magnetic field.[42]

As for the case of a small plasma density, a scaling law for the higher harmonics can be stated:

$$p_{y\ n} \sim k^n\sin(n\phi_\tau) + \mathcal{O}(k^{n+2}), \tag{36a}$$

$$p_{x\ n} \sim k^n\cos(n\phi_\tau) + \mathcal{O}(k^{n+2}). \tag{36b}$$

Again all results can be expressed in terms of a transverse momentum parameter $\hat a$,

$$\hat\eta = \frac{\hat a}{\langle\gamma\rangle}. \tag{37}$$

This time the parameter $\hat a$ is connected with the mean energy by $\langle\gamma\rangle=\sqrt{1+\hat a^2}$.

The dispersion relation follows in addition to the expression for η and ξ from the consistent solution of Eqs. (27a) and (27b):

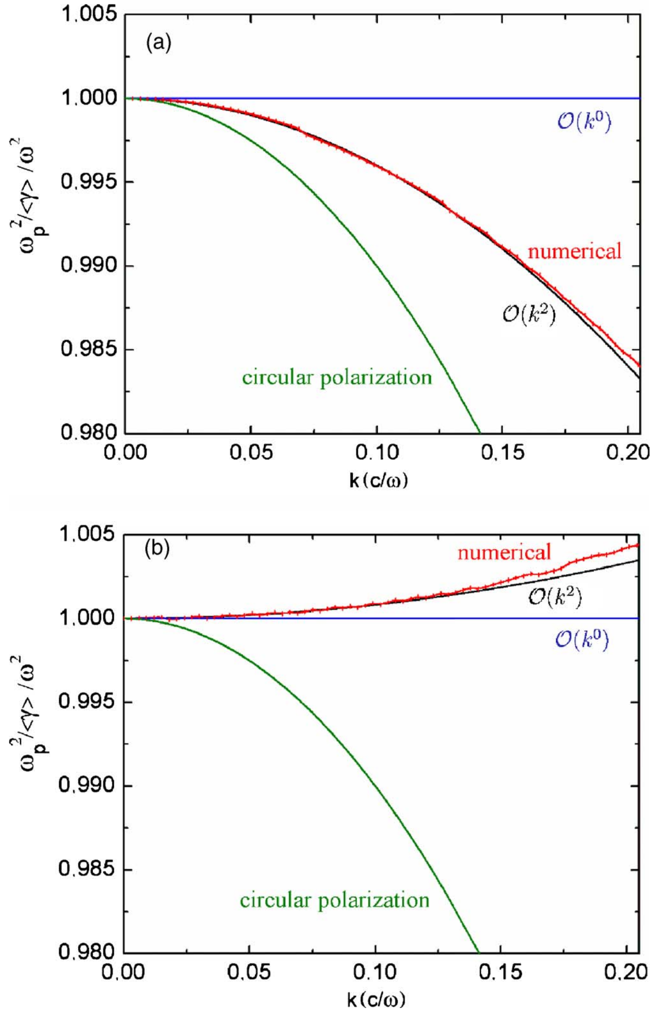

FIG. 6. (Color online) The comparison of the dispersion relation (expressed in the form $\omega_p^2/\langle\gamma\rangle$ as a function of k) for (a) $p_x(0)=-0.6$ and (b) $p_x(0)=-5.0$. Shown are the numerical result, the analytical results up to order k^0 and k^2, and the result for circular polarization.

$$\frac{\omega_p^2}{\langle\gamma\rangle} = \omega^2 - \underbrace{\frac{\hat{\eta}^4 - 6\hat{\eta}^2 + \frac{9}{2}}{(3-\hat{\eta}^2)^2}}_{\substack{\to +\frac{1}{2}(\langle\gamma\rangle\to 1) \\ \to -\frac{1}{8}(\langle\gamma\rangle\to\infty)}} c^2k^2 + \mathcal{O}(k^4). \tag{38}$$

There is an obvious difference to the dispersion relation for circular polarization (23) and also to the dispersion relation of the almost-transverse wave. In the ultrarelativistic case ($\langle\gamma\rangle\to\infty$) the k^2-term has for example the opposite sign.

As illustrated in Fig. 6 for the dispersion relation, all results presented here coincide in the scope of the perturbation theory very well with the results obtained from a numerical integration of the differential equations (7a) and (7b).

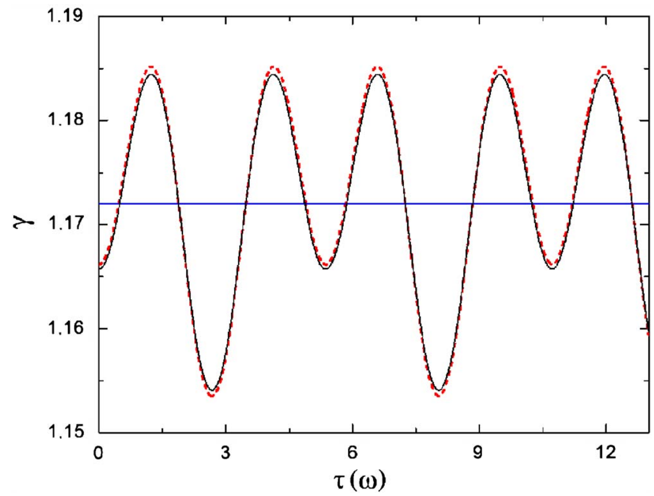

FIG. 7. (Color online) Oscillation of the energy γ about $\langle\gamma\rangle$ for $k=0.1$ and $p_x(0)=-0.6$ (solid: analytical result of order k^2; dashed: numerical result).

A particular interesting quantity in this context is the energy which follows from the solution (33),

$$\gamma = \langle\gamma\rangle\left(1 - k\frac{\hat{\eta}^3}{3-\hat{\eta}^2}\cos(\phi_\tau) + k^2\frac{\hat{\eta}^4 + \frac{9}{2}\hat{\eta}^2 - 9}{(3-\hat{\eta}^2)^2}\cos(2\phi_\tau)\right).$$

For $k=0.1$ and $p_{x0}=-0.6$ the correction of order k^1 $(6\cdot 10^{-3})$ is of the same magnitude as the correction of order k^2 $(-1.2\cdot 10^{-2})$. Both parts oscillate with different frequencies and appear in good agreement in the numerical solution (Fig. 7).

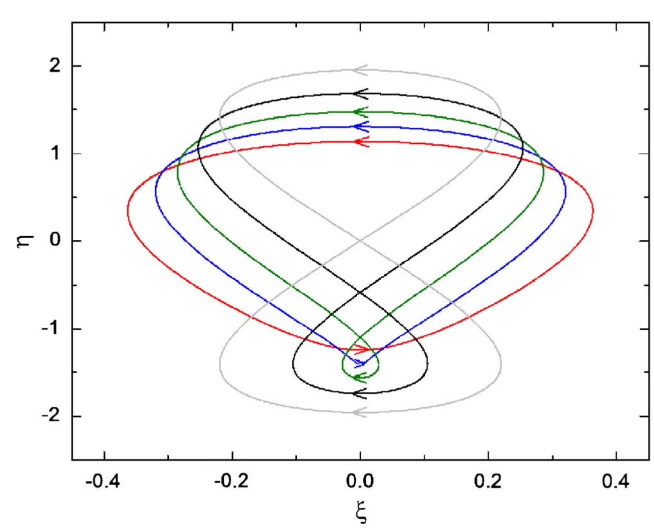

FIG. 8. (Color online) Transition between deformed circular and eight-like trajectories. The trajectories are obtained numerically and correspond to $k=0.38$, 0.4, 0.41, 0.415, and $k=0.42$ (from circle-like to eight-like) calculated for $p_x(0)=-0.6$.

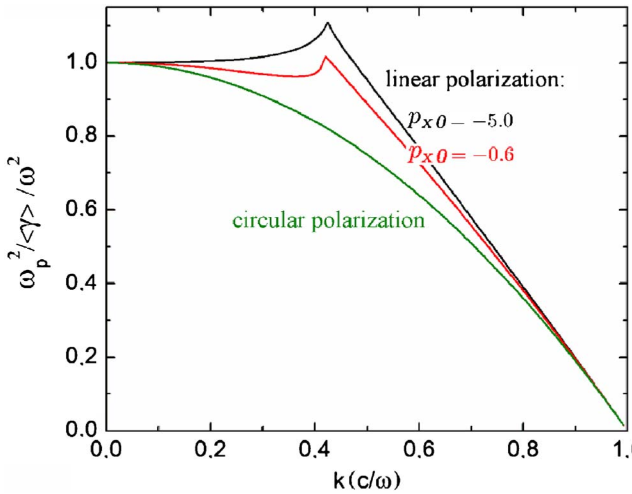

FIG. 9. (Color online) The behavior of the dispersion relation (expressed in the form $\omega_p^2/\langle\gamma\rangle$ as a function of k) during the transition, evaluated for $p_x(0)=-5.0$, $p_x(0)=-0.6$ and the circularly polarized case.

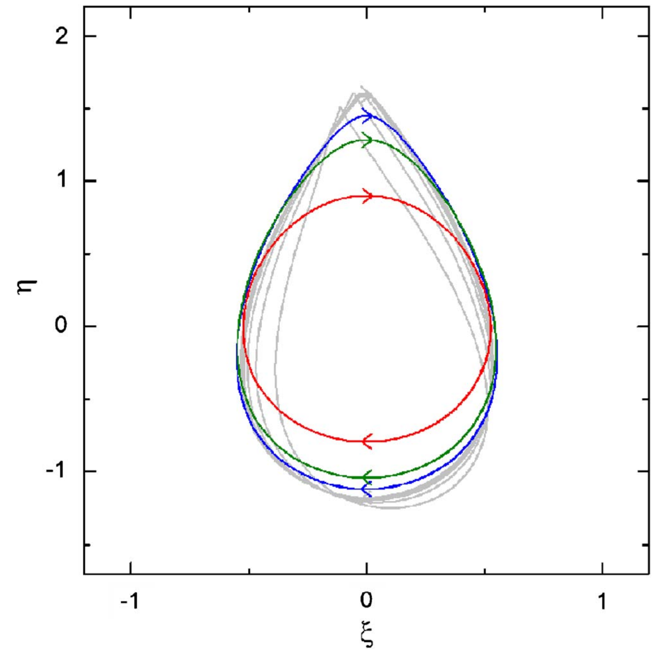

FIG. 10. (Color online) For $p_x(0)>0$ no transition occurs, since no corresponding eight-like solution exist. The trajectories correspond to $k=0.3$, 0.4, 0.42, and 0.43 calculated for $p_x(0)=+0.6$. For $k=0.43$ there exists no longer a closed solution.

V. TRANSITION AT INTERMEDIATE PLASMA DENSITIES

Up to now the two distinct density regimes have been discussed separately. First we have analyzed very small plasma densities and afterwards we have discussed nearly critical plasma densities. Another interesting question concerns the relationship between the small density solution and the nearly critical density solutions. As pointed out above, the transition between the almost-transverse and the vacuum solution seems, because of the common shape, rather straightforward. For the circular solution the relation is unclear and we have therefore to consider intermediate plasma densities. Figure 8 shows the trajectories for various intermediate plasma densities obtained numerically for a fixed initial momentum. One can see the transition from deformed circular trajectories to eight-like trajectories. With increasing k the circular orbit gets more and more deformed. Thereby the ratio of the transverse and the longitudinal amplitude increases. Finally, the orbit develops a loop. After this symmetry breaking the trajectory is a deformed figure-eight and approaches the vacuum solution for asymptotically large k. As shown in Fig. 9 the transition is also reflected in the dispersion relation.

Since no periodic eight-like trajectories exist for $p_x(0)>0$, the transition only occurs for $p_x(0)<0$. For $p_x(0)>0$ an increase in the wavenumber k also leads to more and more deformed circular trajectories until at a critical point closed solutions no longer exist. This behavior is illustrated in Fig. 10.

VI. SUMMARY AND DISCUSSION

In this work, we have studied the propagation of plane electromagnetic waves with a constant phase velocity in cold underdense plasmas at relativistic intensities. Using Lagrangian coordinates of the particles and their proper time, a set of equations for coupled relativistic harmonic oscillators has been derived. These equations are equivalent to the more complicated equations of Akhiezer and Polovin. While previous work was often concerned with circularly polarized waves, the present treatment applies to linear polarization. Two classes of solutions have been analyzed by perturbation expansions in the plasma frequency ω_p and in the refractive index (wavenumber k). The first solution describes eight-like trajectories. The corresponding order ω_p^2 expressions allows one to calculate the completely consistent dispersion relation up to order ω_p^4. The second one, obtained for a nearly critical plasma density, describes circular orbits in the plane of incidence and differs from the commonly considered almost-transverse solution. In addition to the trajectory, we calculated the nontrivial modification of the dispersion relation. A numerical investigation at intermediate plasma densities demonstrates the transition of circular into eight-like solutions. Finally, it is mentioned that collisions were all-over neglected. Their potential contribution to higher order corrections is a different subject beyond the scope of the present work. In future work, it is intended to study in detail the issue of excitation and stability of the solutions, especially for the case of circular orbits.

[1]B. Dromey, M. Zepf, A. Gopal, K. Lancaster, M. S. Wei, K. Krushelnick, M. Tatarakis, N. Vakakis, S. Moustaizis, R. Kodama *et al.*, Nat. Phys. **2**, 456 (2006).

[2]A. J. Mackinnon, M. Borghesi, R. Gaillard, G. Malka, O. Willi, A. A. Offenberger, A. Pukhov, J. Meyer-ter-Vehn, B. Canaud, J. L. Miquel *et al.*, Phys. Plasmas **6**, 2185 (1999).

[3]J. Faure, Y. Glinec, A. Pukhov, S. Kiselev, S. Gordienko, E. Lefebvre, J.-P. Rousseau, F. Burgy, and V. Malka, Nature (London) **431**, 541 (2004).

[4]C. G. R. Geddes, C. Toth, J. van Tilborg, E. Esarey, C. B. Schroeder, D. Bruhwiler, C. Nieter, J. Cary, and W. P. Leemans, Nature (London) **431**, 538 (2004).

[5]S. P. D. Mangles, C. D. Murphy, Z. Najmudin, A. G. R. Thomas, J. L. Collier, A. E. Dangor, E. J. Divall, P. S. Foster, J. G. Gallacher, C. J. Hooker *et al.*, Nature (London) **431**, 535 (2004).
[6]C. E. Max, J. Arons, and A. B. Langdon, Phys. Rev. Lett. **33**, 209 (1974).
[7]X. L. Chen and R. N. Sudan, Phys. Rev. Lett. **70**, 2082 (1993).
[8]J. Faure, Y. Glinec, J. J. Santos, F. Ewald, J.-P. Rousseau, S. Kiselev, A. Pukhov, T. Hosokai, and V. Malka, Phys. Rev. Lett. **95**, 205003 (2005).
[9]T. Brabec and F. Krausz, Rev. Mod. Phys. **72**, 545 (2000).
[10]O. Shorokhov, A. Pukhov, and I. Kostyukov, Phys. Rev. Lett. **91**, 265002 (2003).
[11]A. V. Isanin, S. S. Bulanov, F. F. Kamenets, and F. Pegoraro, Phys. Rev. E **71**, 036404 (2005).
[12]R. Lee and M. Lampe, Phys. Rev. Lett. **31**, 1390 (1973).
[13]A. B. Borisov, O. B. Shiryaev, A. McPherson, K. Boyer, and C. K. Rhodes, Plasma Phys. Controlled Fusion **37**, 569 (1995).
[14]S. V. Bulanov, T. Z. Esirkepov, N. M. Naumova, F. Pegoraro, and V. A. Vshivkov, Phys. Rev. Lett. **82**, 3440 (1999).
[15]V. A. Kozlov, A. G. Litvak, and E. V. Suvorov, Sov. Phys. JETP **49**, 75 (1979).
[16]P. K. Kaw, A. Sen, and T. Katsouleas, Phys. Rev. Lett. **68**, 3172 (1992).
[17]C. Gahn, G. D. Tsakiris, A. Pukhov, J. Meyer-ter-Vehn, G. Pretzler, P. Thirolf, D. Habs, and K. J. Witte, Phys. Rev. Lett. **83**, 4772 (1999).
[18]C. E. Clayton, K. A. Marsh, A. Dyson, M. Everett, A. Lal, W. P. Leemans, R. Williams, and C. Joshi, Phys. Rev. Lett. **70**, 37 (1993).
[19]A. Pukhov and J. Meyer-ter Vehn, Appl. Phys. B **74**, 355 (2002).
[20]M. Schmitz and H.-J. Kull, Europhys. Lett. **58**, 382 (2002).
[21]A. Rousse, K. T. Phuoc, R. Shah, A. Pukhov, E. Lefebvre, V. Malka, S. Kiselev, F. Burgy, J.-P. Rousseau, D. Umstadter *et al.*, Phys. Rev. Lett. **93**, 135005 (2004).
[22]A. I. Akhiezer and R. V. Polovin, Sov. Phys. JETP **3**, 696 (1956).
[23]P. Kaw and J. Dawson, Phys. Fluids **13**, 472 (1970).
[24]W. Lünow, Plasma Phys. **10**, 879 (1968).
[25]C. Max and F. Perkins, Phys. Rev. Lett. **27**, 1342 (1971).
[26]C. E. Max, Phys. Fluids **16**, 1277 (1973).
[27]A. V. Borovsky, A. L. Galkin, V. V. Korobkin, and O. B. Shiryaev, Phys. Rev. E **59**, 2253 (1999).
[28]P. Sprangle, E. Esarey, and A. Ting, Phys. Rev. Lett. **64**, 2011 (1990).
[29]P. Sprangle, E. Esarey, and A. Ting, Phys. Rev. A **41**, 4463 (1990).
[30]E. Esarey, A. Ting, P. Sprangle, D. Umstadter, and X. Liu, IEEE Trans. Plasma Sci. **21**, 95 (1993).
[31]W. Mori, C. Decker, and W. Leemans, IEEE Trans. Plasma Sci. **21**, 110 (1993).
[32]A. I. Zhmoginov and G. M. Fraiman, JETP **100**, 895 (2005).
[33]B. B. Winkles and O. Eldridge, Phys. Fluids **15**, 1790 (1972).
[34]P. C. Clemmow, J. Plasma Phys. **12**, 297 (1974).
[35]A. C.-L. Chian and P. C. Clemmow, J. Plasma Phys. **14**, 505 (1975).
[36]P. C. Clemmow and R. D. Harding, J. Plasma Phys. **23**, 71 (1980).
[37]A. Decoster, Phys. Rep. **47**, 285 (1978).
[38]E. S. Sarachik and G. T. Schappart, Phys. Rev. D **1**, 2738 (1970).
[39]J. M. Rax and N. J. Fisch, Phys. Rev. Lett. **69**, 772 (1992).
[40]P. K. Kaw, A. Sen, and E. J. Valeo, Physica D **9**, 96 (1983).
[41]P. C. Clemmow, J. Plasma Phys. **27**, 267 (1982).
[42]P. C. Clemmow, J. Plasma Phys. **27**, 177 (1982).